Q.1 (a) If \( y = \sin(m \sin^{-1} x) \),

Prove that \( (1 - x^2) \frac{d^2 y}{dx^2} - x \frac{dy}{dx} + m^2 y = 0 \)

Sol. Given: The function \( y = \sin(m \sin^{-1} x) \)  

Differentiating equation (i) w.r.t. \( x \), we get

\[
y_1 = \cos(m \sin^{-1} x) \frac{m}{\sqrt{1-x^2}}
\]

\[
(\sqrt{1-x^2})y_1 = m \cos(m \sin^{-1} x).
\]

Squaring both the sides, we get

\[
(1-x^2)y_1^2 = m^2 \cos^2(m \sin^{-1} x)
\]

\[
(1-x^2)y_1^2 = m^2 [1 - \sin^2(m \sin^{-1} x)]
\]

\[
(1-x^2)y_1^2 = m^2 [1 - y^2] \quad \text{[From equation (i)]}
\]

Differentiating equation (ii) w.r.t. \( x \), we get

\[
(1-x^2)2y_1y_2 - 2xy_1^2 = -m^2 2yy_1
\]

\[
(1-x^2)y_2 - xy_1 = -m^2 y
\]

\[
(1-x^2)y_2 - xy_1 + m^2 y = 0.
\]

Hence Proved.

Q.1 (b) The equation of the tangent at the point \((2, 3)\) of the curve \( y^2 = ax^3 + b \) is \( y = 4x - 5 \). Find the values of \( a \) and \( b \).

Sol. Given: The equation of curve is,

\[
y^2 = ax^3 + b \quad \text{... (i)}
\]

Differentiating equation (i) w.r.t. \( x \), we get

\[
2y \frac{dy}{dx} = 3ax^2
\]

\[
\frac{dy}{dx} = \frac{3ax^2}{2y}
\]

\[
\left( \frac{dy}{dx} \right)_{(2,3)} = \frac{3a(2)^2}{2(3)} = 2a
\]

The equation of tangent at \((2, 3)\) is

\[
(y - y_1) = \left( \frac{dy}{dx} \right)_{(2,3)} (x - x_1)
\]

\[
y - 3 = 2a(x - 2)
\]

\[
y = 2ax - 4a + 3
\]

But given, the equation of tangent is

\[
y = 4x - 5
\]

\[
\text{... (ii)}
\]

Equation (ii) and (iii) represent the same line hence comparing them, we get

\[
1 = \frac{2a}{4} = \frac{-4a + 3}{-5}
\]
$$\frac{2a}{4} = 1 \Rightarrow a = 2 \quad \text{and} \quad -\frac{4a+3}{-5} = 1 \Rightarrow a = 2 .$$

At the point (2, 3), from equation (i), we get

$$3^2 = 2(2)^3 + b \Rightarrow b = 9 - 16 = -7 .$$

\[ \therefore \quad a = 2, b = -7 . \quad \text{Ans.} \]

**Q.1**

(c) Evaluate \( \int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + \cos^4 x} \, dx \).

**Sol.**

Given: \( I = \int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + \cos^4 x} \, dx \)

\[ I = \int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + (1 - \sin^2 x)^2} \, dx . \]

Putting \( \sin^2 x = t \), so that \( 2 \sin x \cos x \, dx = dt \) or \( \sin 2x \, dx = dt \).

When \( x = 0 \) then \( t = 0 \) and when \( x = \frac{\pi}{2} \) then \( t = 1 \).

\[ I = \left[ \frac{1}{2} \int_0^1 \frac{dt}{t^2 - (1-t)^2} \right] \]

\[ I = \left[ \frac{1}{2} \int_0^1 \frac{dt}{t^2 - (1-t)^2} \right] = \frac{1}{2} \left[ \tan^{-1} \left( \frac{t - \frac{1}{2}}{\frac{1}{2}} \right) \right]_0^1 \]

\[ I = \left[ \tan^{-1} \left( \frac{t - \frac{1}{2}}{\frac{1}{2}} \right) \right]_0^1 = \tan^{-1} 1 - \tan^{-1} (-1) \]

\[ I = \frac{\pi}{4} \left( \frac{\pi}{4} \right) \quad \text{Ans.} \]

**Q.2**

(a) Expand by Maclaurin's theorem \( e^{\cos x} \) as far as the term \( x^3 \).

**Sol.**

Given: The function \( y = e^{\cos x} \)

\( \Rightarrow (y)_0 = e^0 = 1 , \)

Differentiating \( y \) w.r.t. \( x \) successively, we get

\( y_1 = e^{\cos x} (1 - \cos x \sin x) = y(\cos x - x \sin x) \quad \Rightarrow (y)_0 = (y)_0 \cdot 1 = 1 , \)

\( y_2 = y_1 (\cos x - x \sin x) + y(- \sin x - x \sin x - x \cos x) \quad \Rightarrow (y)_2 = (y)_2 \cdot 1 = 1 , \)

\( y_3 = y_2 (\cos x - x \sin x) + y_1(- \sin x - \sin x - \sin x - x \cos x) \)

\(- y_2(2 \sin x \cos x) - y(2 \cos x + \cos x \cos x - x \sin x) \quad \Rightarrow (y)_3 = (y)_3 \cdot 1 = 1 , \)

\( y_4 = y_3 (\cos x - x \sin x) + y_2(- 2 \sin x - x \cos x) - 2 y_2 (2 \sin x + x \cos x) \)

\(- 3 y_2 (3 \cos x - x \sin x) - y_1 (3 \cos x - x \sin x) - y(4 \sin x + x \cos x) \quad \Rightarrow (y)_4 = (y)_4 \cdot 3 = 3 , \)

\( y_5 = y_4 (\cos x - x \sin x) - 3 y_3 (2 \sin x + x \cos x) \)

\(- 3 y_3 (3 \cos x - x \sin x) + y(4 \sin x + x \cos x) \quad \Rightarrow (y)_5 = (y)_5 \cdot 3 = 3 , \)

\( y_6 = y_5 (\cos x - x \sin x) - 4 y_4 (2 \sin x + x \cos x) - 6 y_4 (3 \cos x - x \sin x) \)
According to Maclaurin’s series, we have

\[ y = (y_0) + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots + \frac{x^n}{n!}(y_n)_0 + \ldots \]

\[ e^{\cos x} = 1 + \frac{x}{2} - \frac{x^2}{3} - \frac{11x^4}{24} - \frac{x^6}{5} - \ldots \]

Ans.

Q.2 (b) Prove that the curvature at the point \((x, y)\) of the catenary

\[ y = c \cos h \left( \frac{x}{c} \right) \]

is \( y^2 \).

Sol. Given: The curve \( y = c \cosh \left( \frac{x}{c} \right) \)  

Differentiating equation (i) with respect to \( x \), we get

\[ \frac{dy}{dx} = c \sinh \left( \frac{x}{c} \right) \left( \frac{1}{c} \right) = \frac{\sinh \left( \frac{x}{c} \right)}{c} \]

Again differentiating with respect to \( x \), we get

\[ \frac{d^2 y}{dx^2} = \frac{1}{c} \cosh \left( \frac{x}{c} \right) \]

\[ \therefore \text{Radius of curvature} \]

\[ \rho = \frac{\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{3/2}}{\left( \frac{d^2 y}{dx^2} \right)} = \frac{1 + \sinh^2 \left( \frac{x}{c} \right)}{\left( \frac{1}{c} \cosh \left( \frac{x}{c} \right) \right)} \]

\[ \rho = c \cosh^2 \left( \frac{x}{c} \right) = c \left( \frac{y^2}{c} \right) \]  

[Using equation (i)]

\[ \rho = \frac{y^2}{c} \]

Hence Proved.

Q.2 (c) Locate the stationary points of \( x^4 + y^4 - 2x^2 + 4xy - 2y^2 \) and determine their nature.

Sol. Given: \( u = x^4 + y^4 - 2x^2 + 4xy - 2y^2 \)  

For maxima and minima of \( u \), we must have

\[ \frac{\partial u}{\partial x} = 4x^3 - 4x + 4y \]  

[... (ii)]

and

\[ \frac{\partial u}{\partial y} = 4y^3 + 4x - 4y \]  

[... (iii)]

Taking,

\[ \frac{\partial u}{\partial x} = 0 \Rightarrow 4x^3 - 4x + 4y = 0 \]

\[ x^3 - x + y = 0 \]  

[... (iv)]

and

\[ \frac{\partial u}{\partial y} = 0 \Rightarrow 4y^3 + 4x - 4y = 0 \]

\[ y^3 + x - y = 0 \]  

[... (v)]

Adding equation (iv) and (v), we get

\[ x^3 + y^3 = 0 \Rightarrow (x + y)(x^2 - xy + y^2) = 0 \]

\[ x + y = 0 \] but \( x^2 - xy + y^2 \neq 0 \)

\[ x = -y \]  

[... (vi)]
Putting in equation (ii), we get

\[ x^3 - 2x = 0 \]

\[ x = 0, \quad x = \sqrt{2} \]

\[ \therefore y = 0, \quad y = -\sqrt{2}. \quad \text{[From equation (vi)]} \]

Thus the required stationary points are (0,0) and \((\sqrt{2}, -\sqrt{2})\).

Again partially differentiating equation (ii) w.r.t. \( x \) and \( y \), we get

\[ r = \frac{\partial^2 u}{\partial x^2} = 12x^2 - 4 \quad \text{and} \quad s = \frac{\partial^2 u}{\partial x \partial y} = 4. \]

Partially differentiating equation (iii) w.r.t. \( y \), we get

\[ t = \frac{\partial^2 u}{\partial y^2} = 12y^2 - 4. \]

When \( x = \sqrt{2}, \ y = -\sqrt{2} \), we have

\[ r = 12(\sqrt{2})^2 - 4 = 20 > 0, \ s = 4 \quad \text{and} \quad t = 12(\sqrt{2})^2 - 4 = 20 \]

\[ \therefore rt - s^2 = (20)(20) - (4)^2 = 384 > 0. \]

Therefore \( u \) is minimum at \((\sqrt{2}, -\sqrt{2})\).

When \( x = 0, \ y = 0 \), we have

\[ r = 12(0)^2 - 4 = -4 < 0, \ s = 4 \quad \text{and} \quad t = 12(0)^2 - 4 = -4 \]

\[ \therefore rt - s^2 = (-4)(-4) - (4)^2 = 16 - 16 = 0. \]

The condition is doubtful and further investigation is needed.

**Q.3**

(a) If \( u = \sec^{-1}\left(\frac{x^3 - y^3}{x + y}\right) \), then prove that

\[ x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \cot u. \]

**Sol.**

Given: Function \( u = \sec^{-1}\left(\frac{x^3 - y^3}{x + y}\right) \)

\[ \sec u = \frac{x^3 - y^3}{x + y} = \frac{x^3 \left[ 1 - \left(\frac{y}{x}\right)^3 \right]}{x \left[ 1 + \left(\frac{y}{x}\right) \right]} = \frac{x^2 \left[ 1 - \left(\frac{y}{x}\right)^3 \right]}{1 + \left(\frac{y}{x}\right)} \]

which is a homogeneous function of degree 2. Hence by Euler’s theorem we have

\[ x \frac{\partial}{\partial x} (\sec u) + y \frac{\partial}{\partial y} (\sec u) = 2 \sec u \]

\[ x \sec u \cdot \tan u \frac{\partial u}{\partial x} + y \sec u \cdot \tan u \frac{\partial u}{\partial y} = 2 \sec u. \]

Dividing by \( \sec u \cdot \tan u \) on both sides, we get

\[ x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \cot u. \quad \text{Hence Proved.} \]

(b) The radius of a sphere is found to be 10 cm with a possible error of 0.02 cm. What is the relative error in computing the volume?

**Sol.**

Given: \( r = 10 \) cm and \( \delta r = 0.02 \) cm.

\[ \therefore \text{Volume of sphere } V = \frac{4}{3} \pi r^3. \]

Taking log on both sides, we get
\[ \log V = \log \left( \frac{4}{3} \right) + \log \pi + 3 \log r \] 

Differentiating equation (i), we get

\[ \frac{\delta V}{V} = 0 + 0 + 3 \left( \frac{\delta r}{r} \right) \]

\[ \therefore \quad \frac{\delta V}{V} = \text{relative error in } V = 3 \left( \frac{0.02}{10} \right) = 0.006. \]

Thus, relative error in volume of sphere is 0.006. 

Ans.

Q.3 (c) If \( x = r \sin \theta \cos \phi, \ y = r \sin \theta \sin \phi, \ z = r \cos \theta \), then show that \( \frac{\partial (x,y,z)}{\partial (r,\theta,\phi)} = r^2 \sin \theta \).

Sol. Given: Functions \( x = r \sin \theta \cos \phi, \ y = r \sin \theta \sin \phi \) and \( z = r \cos \theta \).

By the definition of Jacobian, we have

\[ \frac{\partial (x,y,z)}{\partial (r,\theta,\phi)} = \begin{vmatrix} \sin \theta \cos \phi & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ r \sin \phi \cos \theta & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ \cos \theta & -r \sin \theta & 0 \end{vmatrix} \]

\[ \frac{\partial (x,y,z)}{\partial (r,\theta,\phi)} = \sin \theta \cos \phi (0 + r^2 \sin^2 \phi \cos \phi) - r \cos \theta \cos \phi (0 - r \sin \theta \cos \phi \sin \phi) + (-r \sin \theta \sin \phi) (-r \sin \theta \cos \phi - r \cos \theta \sin \phi) \]

\[ \frac{\partial (x,y,z)}{\partial (r,\theta,\phi)} = r^2 \sin^3 \theta \cos^2 \phi + r^2 \sin^2 \theta \cos \phi \sin \theta + r^2 \sin \phi \cos \phi \sin \theta \]

Hence Proved.

Q.4 (a) Evaluate \( \lim_{n \to \infty} \left( \frac{1}{1+n^2} + \frac{4}{8+n^2} + \frac{9}{27+n^2} + \ldots + \frac{1}{2n} \right) \)

Sol. Given: \( I = \lim_{n \to \infty} \left( \frac{1}{1+n^2} + \frac{4}{8+n^2} + \frac{9}{27+n^2} + \ldots + \frac{1}{2n} \right) \)

The given series can be written as,

\[ I = \lim_{n \to \infty} \left( \frac{1^2}{1^2+n^2} + \frac{2^2}{2^2+n^2} + \frac{3^2}{3^2+n^2} + \ldots + \frac{n^2}{n^2+n^2} \right) \]

The \( r \)th term of the series is given by,

\[ (r) \text{th} \text{ term} = \frac{r^2}{n^2+r^2}, \text{ where } r \text{ varies from 1 to } n. \]
The required limit of sum \( \lim_{n \to \infty} \sum_{r=1}^{n} \frac{r^2}{n^3 + r^3} \)

\[
= \lim_{n \to \infty} \frac{1}{n^2} \sum_{r=1}^{n} \left( \frac{r^2}{1 + \frac{r^3}{n^3}} \right) = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \left( \frac{r}{n} \right)^2
\]

For the corresponding definite integral, we have

Lower limit = \( \lim_{n \to \infty} \left( \frac{r}{n} \right) \) for the first term

\[
\text{Lower limit} = \lim_{n \to \infty} \left( \frac{1}{n} \right) \quad [\because r = 1 \text{ for the first term}]
\]

i.e.,

Lower limit = 0.

Upper limit = \( \lim_{n \to \infty} \left( \frac{r}{n} \right) \) for the last term

\[
\text{Upper limit} = \lim_{n \to \infty} \left( \frac{n}{n} \right) \quad [\because r = n \text{ for the last term}]
\]

i.e.,

Upper limit = 1.

By summation of series, we get

\[
I = \int_{0}^{1} \frac{x^2}{1 + x^3} \, dx \quad [\because \frac{r}{n} = x \text{ and } \frac{1}{n} = dx]
\]

Putting \( x^3 = t \), so that \( x^2 \, dx = \frac{dt}{3} \)

\[
I = \frac{1}{3} \int_{0}^{1} \frac{dt}{t + 1} = \frac{1}{3} \left[ \log(t + 1) \right]_{0}^{1} = \frac{1}{3} \left[ \log 2 \right] - 0\]

\[
I = \frac{1}{3} \log 2.
\]

Ans.

Q.4 (b) Prove that \( \int_{-\infty}^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{a} \), \( a > 0 \).

Sol. Given:

\[
I = \int_{-\infty}^{\infty} e^{-x^2} \, dx
\]

\[
I = 2 \int_{0}^{\infty} e^{-x^2} \, dx \quad \ldots(i)
\]

Putting \( a^2 \cdot x^2 = y \), i.e., \( x = \frac{\sqrt{y}}{a} \), so that \( dx = \frac{dy}{2a\sqrt{y}} \), from equation (i), we get

\[
I = 2 \int_{0}^{\infty} e^{-y} \frac{1}{2a\sqrt{y}} \, dy
\]

\[
I = \frac{1}{a} \int_{0}^{\infty} e^{-y} y^{-\frac{1}{2}} \, dy
\]

\[
I = \frac{1}{a} \Gamma \left( \frac{1}{2} \right) = \frac{\sqrt{\pi}}{a}
\]

Hence Proved.

Q.4 (c) Express \( \int_{0}^{1} x^n (1-x)^m \, dx \) in terms of beta functions and hence evaluate \( \int_{0}^{1} x^n (1-x)^m \, dx \).
Mathematics - I  
1st Year : Common to all Branches  
Page 7

Sol. Given: \( I = \int_0^1 x^n (1-x^m)^p dx \)

Putting \( x^n = y \) i.e., \( x = y^{\frac{1}{n}} \), so that \( dx = \frac{1}{n} y^{\frac{1}{n}-1} dy \).

When \( x = 0 \) then \( y = 0 \) and when \( x = 1 \) then \( y = 1 \).

\[
\Rightarrow \int_0^1 x^n (1-x^m)^p dx = \frac{1}{n} \int_0^1 y^{\frac{m+1}{n}} (1-y)^{p+1} dy
\]

\[
\int_0^1 x^n (1-x^m)^p dx = \frac{1}{n} \beta\left(\frac{m+1}{n}, p+1\right) \quad \text{Ans.}
\]

Putting \( m = 5, n = 3 \) and \( p = 10 \) in equation (i), we get

\[
\int_0^1 x^5 (1-x^3)^{10} dx = \frac{1}{3} \beta\left(\frac{5+1}{3}, 10+1\right) = \frac{1}{3} \beta(2,11) = \frac{1}{3} \frac{\Gamma(2)\Gamma(11)}{\Gamma(13)} \quad \text{[\because \Gamma(n) = (n-1)!]}\]

\[
\int_0^1 y^3 (1-y^3)^{10} dy = \frac{1}{3} \frac{1!\times10!}{12!} = \frac{1}{3} \frac{10!}{12\cdot11\cdot10!}
\]

\[
\int_0^1 x^3 (1-x^3)^{10} dx = \frac{1}{396}.
\]

Q.5 (a) Evaluate \( \iiint_R y \, dx 
\)

over the part of the plane bounded by the line \( y = x \) and the parabola \( y = 4x-x^2 \).

Sol. Given: \( I = \iiint_R y \, dx \).

The region of integration \( R \) is bounded by curve,

\( y = x \) \hspace{1cm} \text{... (ii)}

and\n
\( y = 4x-x^2 \) \hspace{1cm} \text{... (iii)}

Solving equation (ii) and (iii), we get

\( y = 4x-x^2 \Rightarrow x^2 - 3x = 0 \)

\( x = 0, \ x = 3 \)

\( \Rightarrow \ y = 0, \ y = 3 \) \hspace{1cm} \text{[From equation (ii)]}

Therefore, points of intersection of given curve are \((0, 0)\) and \((3, 3)\).

From the figure \( y \) varies from \( x \) to \( 4x-x^2 \), whereas \( x \) varies from \( 0 \) to \( 3 \).

Hence the given double integral is,

\[
I = \iiint_R y \, dx = \int_0^3 \left[ \int_{x=0}^{4x-x^2} y \, dy \right] \, dx
\]

\[
I = \int_0^3 \left[ \frac{y^2}{2} \right]_{x=0}^{4x-x^2} \, dx
\]

\[
I = \frac{1}{2} \int_0^3 \left[ (4x-x^2)^2 - (0)^2 \right] \, dx
\]

\[
I = \frac{1}{2} \int_0^3 \left[ 16x^2 - 8x^3 + x^4 \right] \, dx
\]

\[
I = \frac{1}{2} \left[ \frac{5x^3 + x^5}{5} - 2x^4 \right]_{x=0}^{x=3}
\]

\[
I = \frac{1}{2} \left[ \frac{405 + 243 - 162}{5} \right]
\]

\[
I = \frac{54}{5}
\]

\( \iiint_R y \, dx = \frac{54}{5} \). \hspace{1cm} \text{Ans.}

Q.5 (b) Evaluate \( \int_0^1 \int_0^{1-x} \int_0^{x+y} xyz \, dz \, dy \, dx \).
Sol. Given:

\[ I = \int_{y=0}^{1} \int_{x=0}^{1-x-y} \int_{z=0}^{1-x-y} xyz \, dz \, dy \, dx \]

\[ I = \int_{y=0}^{1} \int_{x=0}^{1-x-y} \int_{z=0}^{1-x-y} z \, dy \, dx \]

\[ I = \int_{y=0}^{1} \int_{x=0}^{1-x-y} \left( \frac{z^2}{2} \right) \left( 1-x-y \right) \, dy \, dx \]

\[ I = \int_{y=0}^{1} \int_{x=0}^{1-x-y} \left( \frac{(1-x-y)^2}{2} \right) \, dy \, dx \]

\[ I = \frac{1}{2} \int_{x=0}^{1} x \left( \int_{y=0}^{1-x} \left( 1-x-y \right)^2 \, dy \right) \, dx \]

\[ = \frac{1}{2} \int_{x=0}^{1} x \left( \int_{y=0}^{1-x} \left( 1-x \right)^2 - 2(1-x)y + y^2 \right) \, dy \, dx \]

\[ = \frac{1}{2} \int_{x=0}^{1} x \left( \frac{(1-x)^4}{2} - 2 \frac{(1-x)^4}{3} + \frac{(1-x)^4}{4} \right) \, dx \]

\[ = \frac{1}{24} \int_{x=0}^{1} x(1-x)^4 \, dx \]

Putting \( 1-x=t \), so that \( dx = -dt \)

\[ I = \frac{1}{24} \int_{t=1}^{0} (1-t)t^4 (-dt) = \frac{1}{24} \int_{t=0}^{1} (t^4 - t^5) \, dt \]

\[ = \frac{1}{24} \left[ \frac{t^5}{5} - \frac{t^6}{6} \right]_{t=0}^{1} = \frac{1}{24} \left[ \frac{1}{5} - \frac{1}{6} \right] = \frac{1}{24} \left[ \frac{6}{30} \right] = \frac{1}{720} \cdot \text{Ans.} \]

Q.5 (c) Find the area enclosed by the parabolas \( y^2 = 4ax \) and \( x^2 = 4ay \).

Sol. Given: The equations of parabolas are

\[ y^2 = 4ax \quad \text{... (i)} \]

and \[ x^2 = 4ay \quad \text{... (ii)} \]

Squaring both sides in equation (ii), we get

\[ x^4 = 16a^2 y^2 \]

\[ x^4 = 16a^2 (4ax) \]

\[ x(x^3 - 64a^3) = 0 \]

\[ x = 0 \quad \text{and} \quad x^3 = 64a^3 \]

\[ x = 0 \quad \text{and} \quad x = 4a \]

Putting in equation (i), we get

\[ y = 0 \quad \text{and} \quad y = 4a \]

\[ \therefore \] Required point of intersection are (0, 0) and (4a, 4a).

Here

(i) \( y \) varies from \( \frac{x^2}{4a} \) to \( \sqrt{4ax} \).

(ii) \( x \) varies from 0 to 4a.

\[ \therefore \text{Required area is,} \]

\[ A = \iint dx \, dy \]
\[ A = \int_{-a}^{a} \sqrt{4ax - x^2} \, dx \frac{dy}{4a} \]

\[ A = \int_{-a}^{a} \left( \frac{1}{\sqrt{4ax - x^2}} \right) dx = \int_{-a}^{a} \left( \frac{x^2}{4a} \right) dx \]

\[ A = 2 \sqrt{a} \left[ \frac{x^{3/2}}{3} \right]_{0}^{a} - \frac{1}{4a} \left[ \frac{x^3}{3} \right]_{0}^{a} \]

\[ A = \frac{32}{3} a^2 - \frac{16}{3} a^2 \]

The required area is \( A = \frac{16}{3} a^2 \) square units.

**Q.6 (a)** Evaluate \( \int_{b}^{a} e^x \, dx \) as limit of sum.

**Sol.** Given : \( f(x) = e^x \).

We know that by definition of definite integral as limit of sum,

\[ \int_{a}^{b} f(x) \, dx = \lim_{h \to 0} \sum_{i=0}^{n-1} f(a + rh) \]

\[ \int_{a}^{b} e^x \, dx = \lim_{h \to 0} \sum_{i=0}^{n-1} e^{x_i} \]

\[ \int_{a}^{b} e^x \, dx = \lim_{h \to 0} \left[ e^x + e^{x+2h} + e^{x+3h} + \ldots + e^{(n-1)h} \right] \]

\[ \int_{a}^{b} e^x \, dx = \lim_{h \to 0} \left[ e^x + e^{2h} + \ldots + e^{(n-1)h} \right] \]

\[ \int_{a}^{b} e^x \, dx = \lim_{h \to 0} e^x \left[ \frac{1}{(e^h - 1)} \right] \]

\[ \therefore \quad S_n = \frac{a(r^n - 1)}{r - 1}, r > 1 \]

\[ \int_{a}^{b} e^x \, dx = e^x \left( e^{b-a} - 1 \right) \frac{d}{dh} \left( e^x \right) \]

\[ \therefore \quad nh = b - a \]

[Using L’ Hospital’s rule]

\[ \int_{a}^{b} e^x \, dx = (e^b - e^a), \lim_{h \to 0} \frac{1}{e^x} \]

\[ \int_{a}^{b} e^x \, dx = e^b - e^a. \quad \text{Ans.} \]

**Q.6 (b)** Express in terms of the gamma function : \( \int_{0}^{\infty} x^n e^{-x^2} \, dx. \)

**Sol.** Given : \( I = \int_{0}^{\infty} x^n e^{-x^2} \, dx \)

Putting \( x^2 = t, \) i.e., \( x = t^{1/2}, \) so that \( dx = \frac{1}{2} t^{-1/2} \, dt, \) we get

\[ I = \int_{0}^{\infty} x^n e^{-x^2} \, dx = \frac{1}{2} \int_{0}^{\infty} t^{n/2} e^{-t} \, \frac{1}{2} t^{-1/2} \, dt \]

\[ I = \frac{1}{2} \Gamma \left( \frac{n+1}{2} \right) e^{-c/m} \quad \text{where} \quad c = k^2 \text{ and } m = \frac{n+1}{2} \]

\[ I = \frac{1}{2} \Gamma \left( \frac{n+1}{2} \right) e^{-c/m} \]

\[ \therefore \quad \int_{0}^{\infty} e^{-y} \, dy = \frac{\Gamma n}{c^n} \]
\[ I = \frac{\Gamma m}{2k^{2m}} \quad \therefore c = k^2 \]
\[ I = \frac{1}{2k^{n+1}} \Gamma \left( \frac{n+1}{2} \right) \quad \therefore m = \frac{n+1}{2} \]

**Hence Proved.**

**Q.6 (c)** Change the order of integration in \[ \int_0^1 \int_{2-x}^{2-x} xy \, dx \, dy \] and hence evaluate the same.

**Sol.**

Given: \[ I = \int_0^1 \int_{2-x}^{2-x} xy \, dx \, dy \] \( \cdots \) (i)

We draw the bounded region from the given curves:
\[ x = 0, x = 1, y = x^2 \text{ and } y = 2 - x. \]

The possible points for bounded region are: \((0, 0), (1, 1)\) and \((0, 2)\).

On changing the order of integration, integrate first w.r.t. \(x\) by taking two strips parallel to \(x\)-axis say, \(PQ\) and \(RS\).

Limit:
1. \(x\) varies from \(R(x = 0)\) to \(S(x = \sqrt{y})\) and \(y\) varies from \(y = 0\) to \(y = 1\).
2. \(x\) varies from \(P(x = 0)\) to \(Q(x = 2 - y)\) and \(y\) varies from \(y = 0\) to \(y = 2\).

\[ I = \int_{x=0}^{x=1} \int_{y=0}^{y=x^2} xy \, dy \, dx + \int_{x=0}^{x=1} \int_{y=2-x}^{y=2-x} xy \, dy \, dx \]
\[ I = \frac{1}{2} \int_{x=0}^{x=1} y^2 \, dx + \int_{x=0}^{x=1} \left[ \frac{1}{2} \right] y^2 \, dx \]
\[ I = \frac{1}{2} \left[ \frac{y^3}{3} \right]_0^2 + \frac{1}{2} \left[ \frac{4y^3}{3} \right]_0^2 \]
\[ I = \frac{3}{8} \cdot \text{Ans.} \]

**Q.7 (a)** Verify Rolle’s theorem, where \( f(x) = 2x^3 + x^2 - 4x - 2 \).

**Sol.**

Given: The function \( f(x) = 2x^3 + x^2 - 4x - 2 \).

Since a polynomial function is everywhere continuous and differentiable, so the given function is continuous as well as differentiable in every interval.

To identify the interval, we first solve the equation, \( f(x) = 0 \).

\[ 2x^3 + x^2 - 4x - 2 = 0 \]
\[ x^2 (2x+1) - 2 (2x+1) = 0 \]
\[ (x^2 - 2) (2x+1) = 0 \]
\[ x^2 = 2 \quad \text{or} \quad x = -\frac{1}{2} \]
\[ x = \pm \sqrt{2} \quad \text{or} \quad x = -\frac{1}{2}. \]

So, we consider the given function in \([-\sqrt{2}, \sqrt{2}].\)

Clearly, \( f(-\sqrt{2}) = f(\sqrt{2}) = 0 \).

Thus, all the conditions of Rolle’s theorem are satisfied. So there must exist at least one point \( c \in (-\sqrt{2}, \sqrt{2}) \) such that \( f’(c) = 0 \).

But, \( f’(x) = 6x^2 + 2x - 4 \)
Mathematics - I
1st Year : Common to all Branches

Page 11

\[ f'(c) = 0 \Rightarrow 6c^2 + 2c - 4 = 0 \]
\[ 2(3c-2)(c+1) = 0 \]
\[ c = \frac{2}{3} \quad \text{or} \quad c = -1. \]

Clearly, both these points lie in \((-\sqrt{2}, \sqrt{2})\).

Hence, Rolle’s theorem is verified.

Hence Proved.

Q.7 (b) If \( u = f(y-z, z-x, x-y) \), prove that \( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0. \)

Sol. Given: Function \( u = f(y-z, z-x, x-y) \).

Let \( X = y-z, \ Y = z-x \) and \( Z = x-y \) \( \cdots (i) \)

Then \( u = f(X,Y,Z) \), where each one of \( X, Y, Z \) is a function of \( x, y, z \).

Partially differentiating equation \( (i) \) w.r.t. \( x, y \) and \( z \) respectively, we get

\[ \frac{\partial X}{\partial x} = 1, \quad \frac{\partial Y}{\partial x} = 0, \quad \frac{\partial Z}{\partial x} = 1. \]

and

\[ \frac{\partial X}{\partial y} = 0, \quad \frac{\partial Y}{\partial y} = 1, \quad \frac{\partial Z}{\partial y} = 0. \]

Now

\[ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial X} \frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y} \frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z} \frac{\partial Z}{\partial x} \]

\[ \frac{\partial u}{\partial y} = \frac{\partial u}{\partial X} \frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y} \frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z} \frac{\partial Z}{\partial y} \]

\[ \frac{\partial u}{\partial z} = \frac{\partial u}{\partial X} \frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y} \frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z} \frac{\partial Z}{\partial z} \]

Adding equations \( (ii), (iii) \) and \( (iv) \), we get

\[ \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0. \]

Hence Proved.

Q.7 (c) Trace the curve \( y^2(2a-x) = x^3 \).

Sol. Given: The equation of curve

\[ y^2(2a-x) = x^3 \]  
\( \cdots (i) \)

The tracing of curve have following steps:

(i) Symmetry: Here in equation \( (i) \) all power of \( y \) are even, hence the curve is symmetrical about the \( x \)-axis.

(ii) Origin: There is no constant term in this equation. By putting \( x = 0 \), we have \( y = 0 \) the curve passes through the origin.

The tangents at the origin are \( y = 0 \). [Equating to zero the lowest degree terms.]

\[ \therefore \text{Origin is a cusp.} \]

(iii) Points of intersection:

When \( x = 0 \) then \( y = 0 \).
When \( y = 0 \) then \( x = 0 \).

i.e., the curve meets the co-ordinate axis only at origin.

(iv) Asymptotes: Equating coefficient of higher power of \( x \) and \( y \) to 0. We have the asymptotes as follows.

The curve has an asymptote \( x = 2a \) (parallel to \( y \)-axis).

(v) Region: We have, \( y^2 = x^3/(2a-x) \) \( \Rightarrow \ y = \sqrt[3]{x^3/(2a-x)} \).

When \( x \) is –ve, \( y^2 \) is –ve (i.e. \( y \) is imaginary) so that no portion of the curve lies to the left of the \( y \)-axis. Also when \( x > 2a, y^2 \) is again –ve, so that no portion of the curve lies to the right of the line \( 3x = 2a \).

Hence the shape of the curve is as shown in below figure. This curve is known as **cissoids**.