UNIT–01

UNIT-01/LECTURE-01

INTRODUCTION TO COMMUNICATION SYSTEM

Communication:-

Communication is the activity of conveying information through the exchange of ideas, feelings, intentions, attitudes, expectations, perceptions or commands, as by speech, non-verbal gestures, writings, behavior and possibly by other means such as electromagnetic, chemical or physical phenomena and smell. It is the meaningful exchange of information between two or more participants (machines, organisms or their parts).

Communication requires a sender, a message, a medium and a recipient, although the receiver does not have to be present or aware of the sender's intent to communicate at the time of communication; thus communication can occur across vast distances in time and space. Communication requires that the communicating parties share an area of communicative commonality. The communication process is complete once the receiver understands the sender's message.

Communicating with others involves three primary steps:-

1. Thought: First, information exists in the mind of the sender. This can be a concept, idea, information, or feeling.
2. Encoding: Next, a message is sent to a receiver in words or other symbols.
3. Decoding: Lastly, the receiver translates the words or symbols into a concept or information that a person can understand.

The purpose of a Communication System is to transport an information bearing signal from a source to a user destination via a communication channel.

The three basic elements of every communication systems are Transmitter, Receiver and Channel.

Communication engineering is classified into two types based on Transmission media. They are:

1. **Line communication**
2. **Radio communication**

In Line communication the media of transmission is a pair of conductors called transmission line. In this technique signals are directly transmitted through the transmission lines. The installation and maintenance of a transmission line is not only costly and complex, but also overcrowds the open space.

In radio communication transmission media is open space or free space. In this technique signals are transmitted by using antenna through the free space in the form of EM waves.

Block Diagram of Communication System

The communication system consists of three basic components.

1. Transmitter
2. Channel
3. Receiver

1. Transmitter is the equipment which converts physical message, such as sound,words, pictures etc., into corresponding electrical signal.
2. Receiver is equipment which converts electrical signal back to the physical message.
3. Channel may be either transmission line or free space, which provides transmission path between transmitter and receiver.

The Overall purpose of this system is to transfer information from one point (called Source) to another point, the user destination.

The message produced by a source, normally, is not electrical. Hence an input transducer is used for converting the message to a time varying electrical quantity called message signal. Similarly, at the destination point, another transducer converts the electrical waveform to the appropriate message.

The transmitter is located at one point in space, the receiver is located at some other point separate from the transmitter, and the channel is the medium that provides the electrical connection between them.

The purpose of the transmitter is to transform the message signal produced by the source of information into a form suitable for transmission over the channel.

The received signal is normally corrupted version of the transmitted signal, which is due to channel imperfections, noise and interference from other sources.

The receiver has the task of operating on the received signal so as to reconstruct a recognizable form of the original message signal and to deliver it to the user destination.

Communication Systems are divided into 3 categories:

1. Analog Communication Systems are designed to transmit analog information using analog modulation methods.
2. Digital Communication Systems are designed for transmitting digital information using digital modulation schemes, and
3. Hybrid Systems that use digital modulation schemes for transmitting sampled and quantized values of an analog message signal.

Analog Communication:-

Analog Communication is a data transmitting technique in a format that utilizes continuous signals to transmit data including voice, image, video, electrons etc. An analog signal is a variable signal continuous in both time and amplitude which is generally carried by use of modulation.

Analog circuits do not involve quantisation of information unlike the digital circuits and consequently have a primary disadvantage of random variation and signal degradation, particularly resulting in adding noise to the audio or video quality over a distance.

Data is represented by physical quantities that are added or removed to alter data. Analog transmission is inexpensive and enables information to be transmitted from point-to-point or from one point to many. Once the data has arrived at the receiving end, it is converted back into digital form so that it can be processed by the receiving computer.

Analog Communication System

Analog communication is that types of communication in which the message or information signal i.e transmitted is analog in nature. This means that in analog communication the modulating signal (i.e base-band signal) is an analog signal. This analog message signal may be obtained from sources such as speech, video shooting etc.

Advantages:-

1. More easy to generate.

2. Easy way of communication.

Disadvantages:-

1. Very difficulty to transmit as it is.
2. Devices used are expensive.
3. Lots and lots of noise interruptions.
4. Accuracy is less.
5. Transmission and reception is not very easy.

Digital Communication System:-

In digital communication, the message signal to be transmitted is digital in nature. This means that digital communication involves the transmission of information in digital form.

Elements of digital communication systems

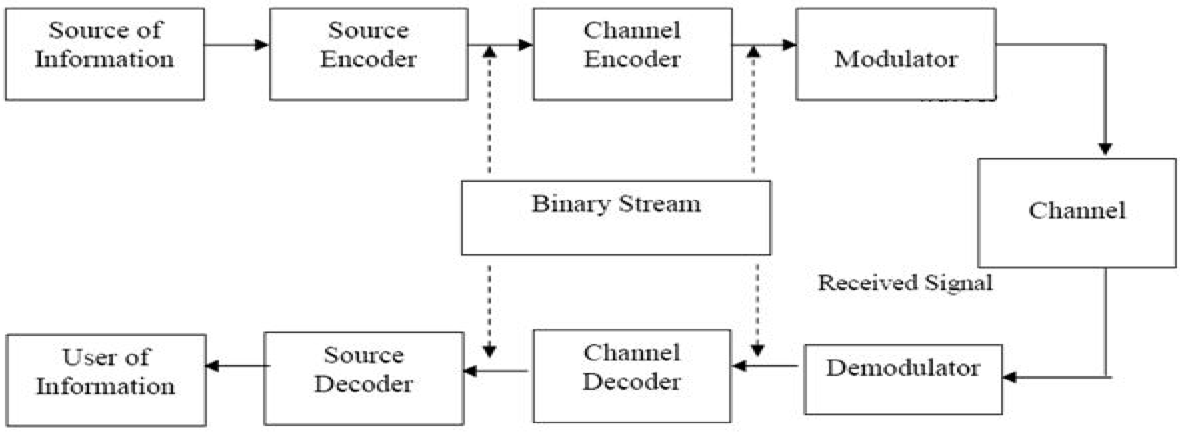
The figure shows the functional elements of a digital communication system. Source of Information:

1. Analog Information Sources.
2. Digital Information Sources.

Analog Information Sources→ Microphone actuated by a speech, TV Camera scanning a scene, continuous amplitude signals.

Digital Information Sources→ These are teletype or the numerical output of computer which consists of a sequence of discrete symbols or letters. An Analog information is transformed into a discrete information through the process of sampling and quantizing.

Digital Communication System



Block Diagram Of Digital Communication System

Source encoder / decoder

The Source encoder (or Source coder) converts the input i.e. symbol sequence into a binary sequence of 0's and 1's by assigning code words to the symbols in the input sequence. For eg. If a source set is having hundred symbols, then the number of bits used to represent each symbol will be 7 because 27=128 unique combinations are available. The important parameters of a source encoder are block size, code word lengths, average data rate and the efficiency of the coder (i.e. actual output data rate compared to the minimum achievable rate)

At the receiver, the source decoder converts the binary output of the channel decoder into a symbol sequence. The decoder for a system using fixed–length code words is quite simple, but the decoder for a system using variable–length code words will be very complex. Aim of the source coding is to remove the redundancy in the transmitting information, so that bandwidth required for transmission is minimized. Based on the probability of the symbol code word is assigned. Higher the probability, shorter is the codeword.

Ex: Huffman coding.

Channel encoder / decoder

Error control is accomplished by the channel coding operation that consists of systematically adding extra bits to the output of the source coder. These extra bits do not convey any information but helps the receiver to detect and / or correct some of the errors in the information bearing bits.

There are two methods of channel coding:

Block Coding:

The encoder takes a block of ‘k’ information bits from the source encoder and adds ‘r’ error control bits, where ‘r’ is dependent on ‘k’ and error control capabilities desired.

Convolution Coding:

The information bearing message stream is encoded in acontinuous fashion by continuously interleaving information bits and error control bits.

The Channel decoder recovers the information bearing bits from the coded binary stream. Error detection and possible correction is also performed by the channel decoder.

The important parameters of coder / decoder are: Method of coding, efficiency, error control capabilities and complexity of the circuit.

Modulator

The Modulator converts the input bit stream into an electrical waveform suitable for transmission over the communication channel. Modulator can be effectively used to minimize the effects of channel noise, to match the frequency spectrum of transmitted signal with channel characteristics, to provide the capability to multiplex many signals.

Demodulator

The extraction of the message from the information bearing waveform produced by the modulation is accomplished by the demodulator. The output of the demodulator is bit stream. The important parameter is the method of demodulation.

Channel

The Channel provides the electrical connection between the source and destination. The different channels are: Pair of wires, Coaxial cable, Optical fibre, Radio channel, Satellite channel or combination of any of these. The communication channels have only finite Bandwidth, non-ideal frequency response, the signal often suffers amplitude and phase distortion as it travels over the channel. Also, the signal power decreases due to the attenuation of the channel. The signal is corrupted by unwanted, unpredictable electrical signals referred to as noise.

The important parameters of the channel are Signal to Noise power Ratio (SNR), usable bandwidth, amplitude and phase response and the statistical properties of noise.

Advantages of Digital Communication:-

1. The effect of distortion, noise and interference is less in a digital communication system. This is because the disturbance must be large enough to change the pulse from one state to the other.

2. Regenerative repeaters can be used at fixed distance along the link, to identify and regenerate a pulse before it is degraded to an am biguous state.

3. Digital circuits are more reliable and cheaper compared to analog circuits.

4. The Hardware implementation is more flexible than analog hardware because of the use of microprocessors, VLSI chips etc.

5. Signal processing functions like encrypti on, compression can be employed to maintain the secrecy of the information.

6. Error detecting and Error correcting codes improve the system performance by reducing the probability of error.
7. Combining digital signals using TDM is simpler than combining analog signals using FDM. The different types of signals such as data, telephone, TV can be treated as identical signals in transmission and switching in a digital communication system.
8. We can avoid signal jamming using spread spectrum technique.

Disadvantages of Digital Communication:-

1. Large System Bandwidth:-Digital transmission requires a large system bandwidth to communicate the same information in a digital format as compared to analog format.
2. System Synchronization:-Digital detection requires system Synchronization whereas the analog signals generally have no such requirement.

Signals:

A signal is a function representing a physical quantity, and typically it contains information about the behavior or nature of the phenomenon. From a communication point of view a signal is any function that carries some information.

Classification of Signals:-

Even and Odd Signals

We say that a continuous signal x(t) is even if x(t)=x(−t) for all t. Similarly, x(t) is odd if x(t)=−x(−t) for all t.

Note that if x(t) is odd, x(0)=0.

Some common even signals you will be familiar with are x(t)=cos(t) and $x(t) = t^2$. Some common odd signals you will be familiar with are x(t)=sin(t) and $x(t) = t^3$.

Periodic and Nonperiodic Signals

A periodic signal is a signal x(t) that satisfies the property x(t) = x(t+kT0) for all t, and all integers k. T0 is called the period of signal.

Energy and Power Signals

If v(t) and i(t) are respectively, the voltage and current across a resistor with resistance R=1Ω resstor, then the instantaneous power is The average energy expended over the time intervalat

$$p(t) = v(t).i(t) = \frac{1}{R}v^2(t) = v^2(t)$$

$$Ex = \frac{1}{T}\int_{t1}^{t2} p(t)\,dt = \frac{1}{T}\int_{t1}^{t2} v^2(t)\,dt \qquad t_1 \leq t \geq t_2$$

For any signal x(t), the energy E_x is defined as

$$E_x = \lim_{T\to\infty}\int_{-T}^{T} |x(t)|^2\,dt = \int_{-\infty}^{\infty} |x(t)|^2\,dt$$

The power Px of signal as

$$P_x = \lim_{T\to\infty}\int_{-T}^{T} |x(t)|^2\,dt$$

For real signals $|x(t)|^2 = x^2(t)$.

The signal x(t) is energy- type signal if and only if Ex well defined and finite $0 < E_x > \infty$. For power-type signal $0 < P_x > \infty$.

Continuous signal:-

A continuous signal or a continuous-time signal is a varying quantity (a signal) whose domain, which is often time, is a continuum (e.g., a connected interval of the reals). That is, the

function's domain is an uncountable set. The function itself need not be continuous.
A signal of continuous amplitude and time is known as a continuous time signal or an analog signal. This (a signal) will have some value at every instant of time. The electrical signals derived in proportion with the physical quantities such as temperature, pressure, sound etc. are generally continuous signals. The other examples of continuous signals are sine wave, cosine wave, triangular wave etc.

Discrete-time signal:-

A discrete signal or discrete-time signal is a time series consisting of a sequence of quantities. In other words, it is a time series that is a function over a domain of integers.
Unlike a continuous-time signal, a discrete-time signal is not a function of a continuous argument; however, it may have been obtained by sampling from a continuous-time signal, and then each value in the sequence is called a sample. When a discrete-time signal obtained by sampling a sequence corresponds to uniformly spaced times, it has an associated sampling rate ; the sampling rate is not apparent in the data sequence, and so needs to be associated as a characteristic unit of the system.

Analog:-

Analog signals are continuous in both time and value. Analog signals are used in many systems, although the use of analog signals has declined with the advent of cheap digital signals. All natural signals are Analog in nature.
Analog systems are less tolerant to noise, make good use of bandwidth, and are easy to manipulate mathematically. However, analog signals require hardware receivers and transmitters that are designed to perfectly fit the particular transmission. If you are working on a new system, and you decide to change your analog signal, you need to completely change your transmitters and receivers.

Digital:-

Digital signals are discrete in time and value. Digital signals are signals that are represented by binary numbers, "1" or "0". The 1 and 0 values can correspond to different discrete voltage values, and any signal that doesn't quite fit into the scheme just gets rounded off.
Digital signals are more tolerant to noise, but digital signals can be completely corrupted in the presence of excess noise. In digital signals, noise could cause a 1 to be interpreted as a 0 and vice versa, which makes the received data different than the original data. Imagine if the army transmitted a position coordinate to a missile digitally, and a single bit was received in error? This single bit error could cause a missile to miss its target by miles. Luckily, there are systems in place to prevent this sort of scenario, such as checksums and CRCs, which tell the receiver when a bit has been corrupted and ask the transmitter to resend the data. The primary benefit of digital signals is that they can be handled by simple, standardized receivers and transmitters, and the signal can be then dealt with in software (which is comparatively cheap to change).

Comparison Between Analog and digital signals:-

Analog and digital signals are used to transmit information, usually through electric signals. In both these technologies, the information, such as any audio or video, is transformed into electric signals. The difference between analog and digital technologies is that in analog technology, information is translated into electric pulses of varying amplitude. In digital technology, translation of information is into binary format (zero or one) where each bit is representative of two distinct amplitudes.

Comparison chart

	Analog	Digital

Signal	Analog signal is a continuous signal which represents physical measurements.	Digital signals are discrete time signals generated by digital modulation
Waves	Denoted by sine waves	Denoted by square waves
Example	Human voice in air, analog electronic devices.	Computers, CDs, DVDs, and other digital electronic devices.
uses	Can be used in analog devices only. Best suited for audio and video transmission.	Best suited for Computing and digital electronics.
Applications	Thermometer	PCs, PDAs
Bandwidth	Analog signal processing can be done in real time and consumes less bandwidth.	There is no guarantee that digital signal processing can be done in real time and consumes more bandwidth to carry out the same information.
cost	Low cost and portable	Cost is high and not easily portable

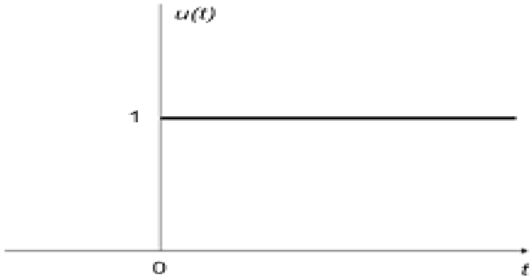
Functions:-

Unit Step Function

Defination: The unit step function, u(t), is defined as

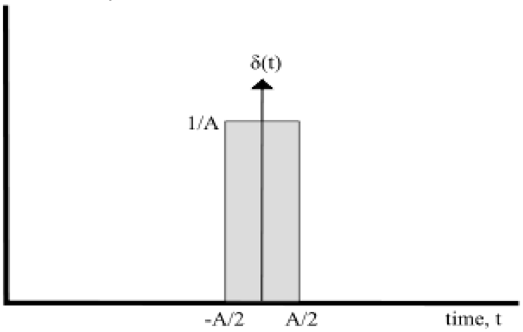
$$u(t)=\begin{cases} 1 & t>0 \\ 0 & t<0 \end{cases}$$

That is, u is a function of time t, and u has value zero when time is negative (before we flip the switch); and value one when time is positive (from when we flip the switch).



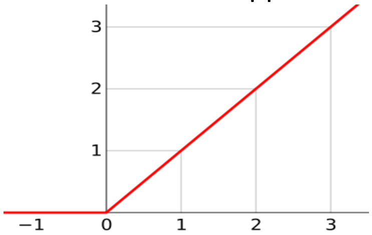
Impulse function

An impulse function is a special function that is often used by engineers to model certain events. An impulse function is not realizable, in that by definition the output of an impulse function is infinity at certain values. An impulse function is also known as a "delta function", although there are different types of delta functions that each have slightly different properties. Specifically, this unit-impulse function is known as the Dirac delta function.



Ramp function

The ramp function is a unary real function, easily computable as the mean of the independent variable and its absolute value. This function is applied in engineering (e.g., in the theory of DSP). The name ramp function is derived from the appearance of its graph.



Sampling function

The sampling function is defined by $Sa(x) = \frac{sin\ x}{x}$

Closely related to the sampling function is the Sinc function defined by $Sinc\ x = \frac{\sin \pi x}{\pi x}$

Time/Frequency Domain Representation of Signals:-

Time domain analysis

Time domain analysis is analyzing the data over a time period. Functions such as electronic signals, market behaviors, and biological systems are some of the functions that are analyzed using time domain analysis. For an electronic signal, the time domain analysis is mainly based on the voltage – time plot or the current – time plot. In a time domain analysis, the variable is always measured against time. There are several devices used to analyze data on a time domain basis. The cathode ray oscilloscope (CRO) is the most common device when analyzing electrical signals on a time domain.

Frequency domain analysis

Frequency domain is a mthod used to analyze data. This refers to analyzing a mathematical function or a signal with respect to the frequency. Frequency domain analysis is widely used in fields such as control systems engineering, electronics and statistics. Frequency domain analysis is mostly used to signals or functions that are periodic over time. This does not mean that frequency domain analysis cannot be used in signals that are not periodic.

The most important concept in the frequency domain analysis is the transformation. Transformation is used to convert a time domain function to a frequency domain function and vice versa. The most common transformation used in the frequency domain is the Fourier transformations. Fourier transformation is used to convert a signal of any shape into a sum of infinite number of sinusoidal waves. Since analyzing sinusoidal functions is easier than analyzing general shaped functions, this method is very useful and widely used.

A signal f(t) can be represented in terms of relative amplitude of various frequency components present in signal. This is possible by using exponential Fourier series. This is a frequency domain representation of the signal. The time domain representation specifies a signal value at each instant of time. This means that a signal f(t) can be specified in two equivalent ways:

1. Time domain representation; where f(t) is represented as a function of time. Graphical time domain representation is termed as waveform.
2. he frequency domain representation; where the signal is represented graphical in terms of its spectrum.

Any of the above two representation uniquely specifies the function, i.e. if the signal f(t) is specified in time domain, we can determine its spectrum. Conversely, if the spectrum is specified, we can determine the corresponding time domain of signal. In order to determine the function, it is necessary that both amplitude spectrum and phase spectrum are specified. However, in many cases, the spectrum is either real or imaginary, as such, only an amplitude plot is enough as all frequency component have identical phase relation.

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FOURIER TRANSFORM

There are a number of different mathematical transforms which are used to analyze time functions and are referred to as frequency domain methods. The following are some most common transforms, and the fields in which they are used:

- Fourier series – repetitive signals, oscillating systems
- Fourier transform – nonrepetitive signals, transients
- Laplace transform – electronic circuits and control systems
- Z transform – discrete signals, digital signal processing

Signals can be transformed between the time and the frequency domain through various transforms. The signals can be processed within these domains and each process in one domain has a corollary in the other.

The Fourier transform, named for Joseph Fourier, is a mathematical transform that expresses a mathematical function of time as a function of frequency. For instance, the transform of a musical chord made up of pure notes without overtones, expressed as loudness as a function of time, is a mathematical representation of the amplitudes and phases of the individual notes that make it up. The function of time is often called the time domain representation, and the function of frequency is called the frequency domain representation. The inverse Fourier transform expresses a frequency domain function in the time domain. Each value of the function is usually expressed as a complex number (called complex amplitude) that can be interpreted as an absolute value and a phase component. In the case of a periodic function, such as a musical tone (possibly with overtones), the Fourier transform can be simplified to the calculation of a discrete set of complex amplitudes, called Fourier series coefficients.

The Fourier transform has many applications in physics and engineering. Fourier transformation from the time domain to the frequency domain transforms differential equations into algebraic equations and convolution into multiplication. This often results in simplification of needed mathematical manipulations. The Fourier transform is reversible, being able to transform from either domain to the other. The term itself refers to both the transform operation and to the function it produces.

So far, the discussion was confined to the use of Fourier series in the analysis of the following cases of waveform;

An arbitrary waveform over a finite interval,

A periodiv waveform over an entire interval $(-\infty, \infty)$.

However, it is desirable to analyse any general waveform, periodic or not, over an entire interval $(-\infty, \infty)$. Because a vast majority of interesting signal extend for all time

$(-\infty, \infty)$. And are non periodic in nature.

<u>Merits of Fourier Transform:-</u>

A transform is a set of rules substituting one function for another. A function f(t) can have a variety of transforms. Fourier transform is most useful tool for analysing signal involved in communication systems. Some of the main advantages of this transform are.

1. The original time functions can be uniquely recovered from in.
2. It has a property analogous to common logrithm that helps in evaluating convolution

integrals

3. Although Laplace transforms is extensively used for solving the problems of electrical systems, Fourier transform is much more useful in communication systems because here the phase and amplitude characteristics are readily known. Laplace transform, on the other hand, is more useful in electrical systems, where network transfer function as a ratio of polynomials in s is readily specified and the analysis is based on pole and zeros.

Limitation of Fourier Transform:-

The Fourier transform $(s = j\omega)$is closely related with Laplace transform. In Fourier transform, damping factor$\sigma = 0$, and hence the Fourier transform may not converge for many time functions. In other words, there are many time function for which Fourier transform does not exist. Such functions are not absolutely integrable, i.e. their Fourier integral dose not converse in the limit$T \rightarrow \infty$. The Laplace transform for such function may exist because damping factor σ makes the Laplace integral to converge.

Exitence of Fourier Transform:-

For a function f(t) to be Fourier transformable, it is sufficient that f(t) satisfy Dirichlet's conditions given below:

The function f(t) is a single-valued with a finite number of maxima and minima; and a finite number of discontinuities in any finite time interval.

The function f(t) is absolutely integrable , i.e.

$$\int_{-\infty}^{\infty} |f(t)|dt < \infty$$

Fourier Transform:-

$$\mathbf{F(\omega) = F[\ f(t)]}$$

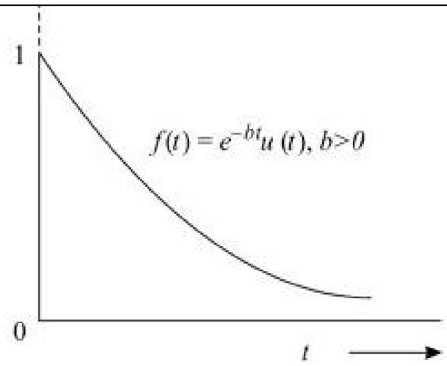
$$\mathbf{f(t) = F^{-}[F(\omega)]}$$

$$\boldsymbol{FT[f(t)] = F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\,\omega t} dt}$$

Inverse Fourier Transforms:-

$$\boldsymbol{F^{-}\ [F(\omega)] = f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega}$$

Example 1. Find the Fourier transform of a single-sided exponential function $e^{-bt}\, u(t)$. Shown in fig. and draw the spectrum.



Solution:

$$f(t) = e^{-bt}\, u(t)$$

$$F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} dt$$

$$= \int_{-\infty}^{\infty} e^{-bt} u(t) e^{-j\omega t} dt$$

Value of Unit step function from -∞ to 0 is zero. And 0 to +∞ is one.

$$F(\omega) = \int_{0}^{\infty} e^{-bt} e^{-j\omega t} dt = \int_{0}^{\infty} e^{-(b+j\omega)t} dt$$

$$= \frac{1}{b+j\omega} = \frac{(b-j\omega)}{(b+j\omega)(b-j\omega)} = \frac{b}{b^2+\omega^2} - j\frac{\omega}{b^2+\omega^2}$$

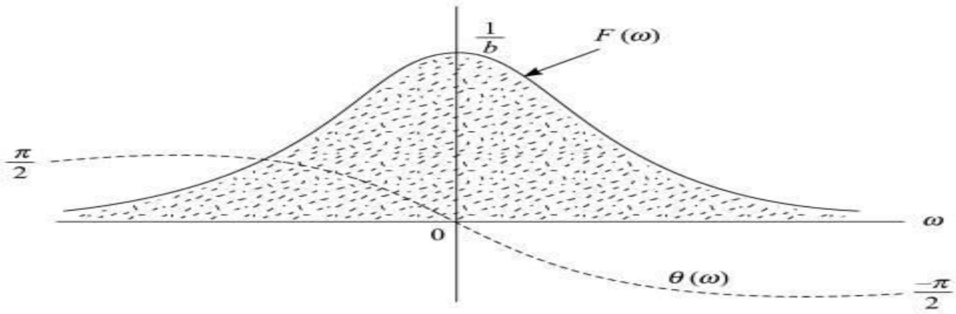
$$= \frac{1}{\sqrt{b^2+\omega^2}} e^{-j\tan^{-1}\left(\frac{\omega}{b}\right)}$$

The magnitude spectrum $|F(\omega)|$ and phase spectrum $\theta(\omega)$ are given as.

$$|F(\omega)| = \frac{1}{\sqrt{b^2+\omega^2}}$$

$$\theta(\omega) = -\tan^{-1}\left(\frac{\omega}{b}\right)$$

The magnitude and phase spectrum are shown in fig. The fourier transform exists only for +ve value.



Spe ctrum of single sided exponential function

Example 2. Find the Fourier transform of a double-sided exponential function$e^{-b|t|}\, u(t)$.

Shown in fig. and draw the spectrum. **(JUNE 2010)(10)**

Solution: The Fourier transform of a double-sided exponential function $e^{-b|t|}\, u(t)$ is

$$f(t) = e^{-b|t|}$$

$$F(\omega) = \int_{-\infty}^{\infty} e^{-b|t|} e^{-j\omega t} dt$$

$$F(\omega) = \int_{-\infty}^{0} e^{-b(-t)} e^{-j\omega t} dt + \int_{0}^{\infty} e^{-bt} e^{-j\omega t} dt$$

$$F(\omega) = \int_{-\infty}^{0} e^{(b-j\omega)t} dt + \int_{0}^{\infty} e^{-(b+j\omega)t} dt$$

$$= \frac{1}{b-j\omega} + \frac{1}{b+j\omega} = \frac{2b}{b^2+\omega^2}$$

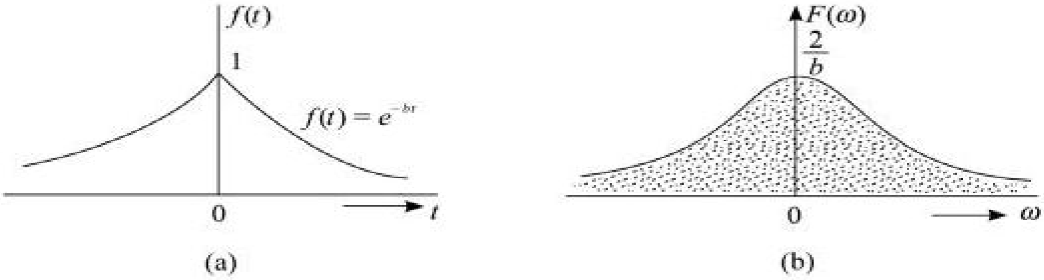


Fig. (a) Time domain of double-sided exponential function and fig. (b) Frequency domain of fig.(a).

Example 3. Find the Fourier transform of Gate function of amplitude K and width τ as shown

in fig. (a).

Solution: A Gate function $G\tau(t)$ is rectangular pulse defined by

$$f(t) = G_\tau(t) = \begin{cases} K & |t| < \frac{\tau}{2}; \\ 0 & \text{elsewhere} \end{cases}$$

The Fourier transform of this function is obtained as follows.

$$F(\omega) = \int_{-\tau/2}^{\tau/2} K e^{-j\omega t} dt = \frac{K}{j\omega}\left(e^{\frac{j\omega\tau}{2}} - e^{\frac{-j\omega\tau}{2}}\right)$$

$$= \frac{2K}{\omega}\left(\frac{e^{\frac{j\omega\tau}{2}} - e^{\frac{-j\omega\tau}{2}}}{2j}\right)$$

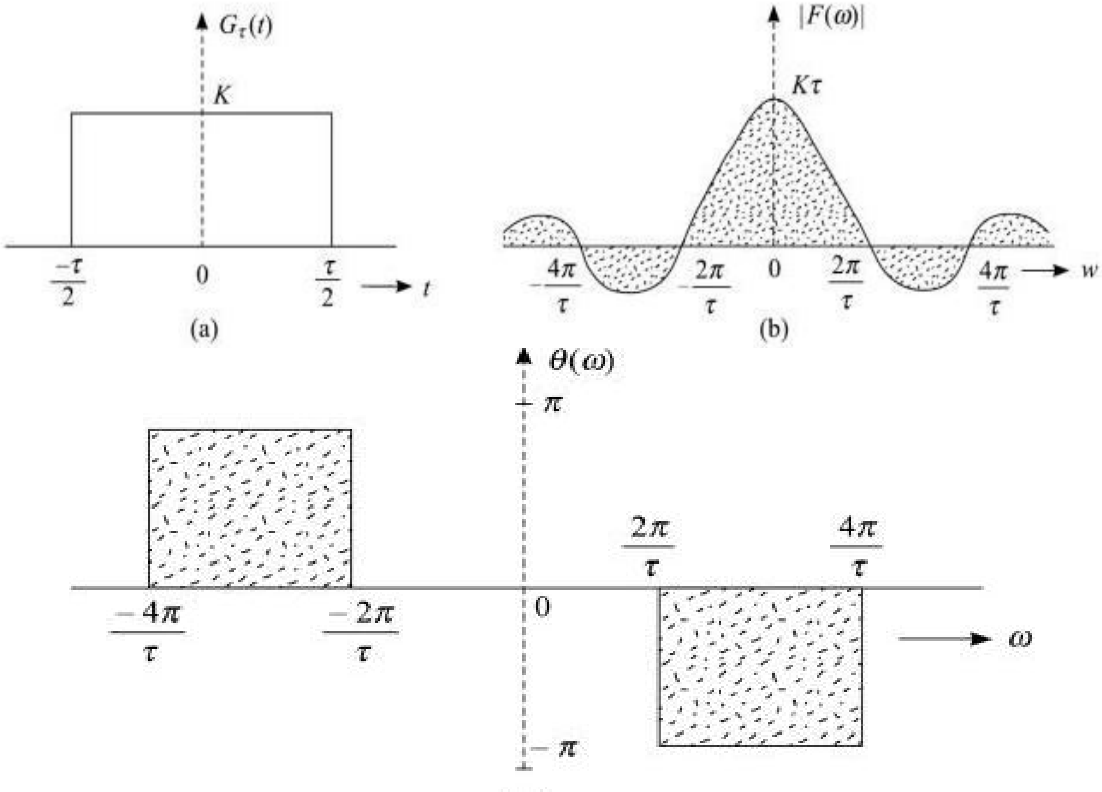
$$= \frac{K}{\omega/2} \sin\left(\frac{\omega\tau}{2}\right)$$

Multiplying and dividend by τ

$$F(\omega) = K\tau \frac{\sin\left(\omega \frac{\tau}{2}\right)}{\left(\omega \frac{\tau}{2}\right)} = K\tau\ Sa\left(\frac{\omega\tau}{2}\right)$$

Where k is amplitude and τ is duration.

The amplitude and phase spectrum are shown in fig (b) and (c) respectively.



(c)

Example 4. Find the Fourier transform of impulse function. $f(t) = \delta(t)$.

Solution: The Fourier transform of impulse function is.

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt$$

Using shifting property of impulse function is

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

$$\int_{-\infty}^{\infty} f(t)\,\delta(t)dt = f(0)\int_{-\infty}^{\infty} \delta(t)\,dt = f(0).1 = f(0)$$

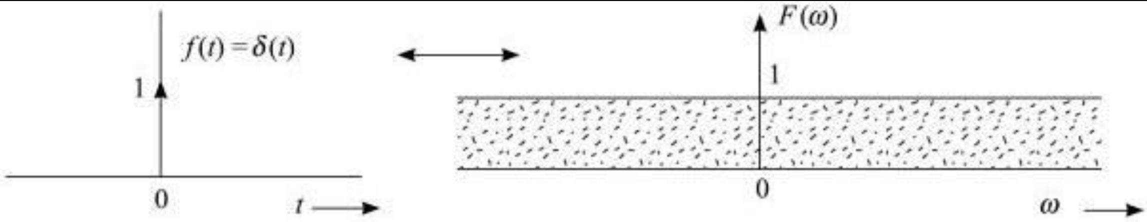
$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

so

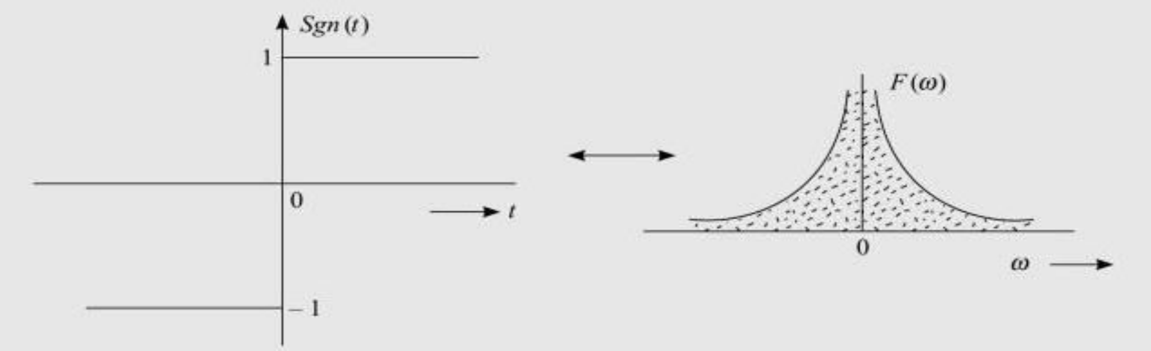
$$F(\omega) = \left[e^{-j\omega t}\right]_{t=0} = 1$$

$\delta(t) \leftrightarrow 1$

The Fourier transform of an impulse function is unity.



Example 5. Find the Fourier transform of Signum function denoted by sgn(t).



Sulution: The Signum function shown in Fig. is expressed as.

$$Sgn(t) = \begin{cases} 1, & t > 0 \\ -1, & t < 0 \end{cases}$$

Signum function in turns of unit step function.

$$Sgn(t) = 2u(t) - 1$$

Or

$Sgn(t) = u(t) - u(-t)$ ....(1)

$$\lim_{a \to 0} [e^{-at}u(t)] = u(t)]$$

and

$\lim_{a \to 0}[e^{at}u(-t)] = u(-t)]$

So eqn no. 1 is

$$Sgn(t) = \lim_{a \to 0}\left[e^{-at}u(t) - e^{at}u(-t)\right]$$

$$F[Sgn(t)] = \lim_{a \to 0}\left[\int_0^{\infty} e^{-at}e^{-j\omega t}dt - \int_{-\infty}^{0} e^{at}e^{-j\omega t}dt\right]$$

$$= \lim_{a \to 0}\left[\frac{-2j\omega}{a^2 + \omega^2}\right]$$

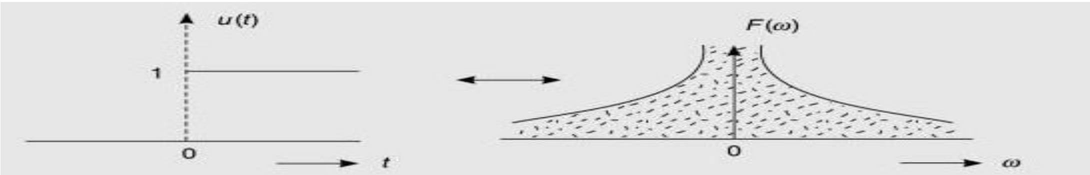
$$= \frac{2}{j\omega}$$

	RGPV QUESTIONS	Year	Marks
Q.1	Find the Fourier transform of a single-side exponential function $e^{-bt}u(t)$ also draw the spectrum, where u(t) is unit step function	JUNE 2010	10

UNIT-01/LECTURE-03

NUMERICAL AND PROPERTY OF FOURIER TRANSFORM

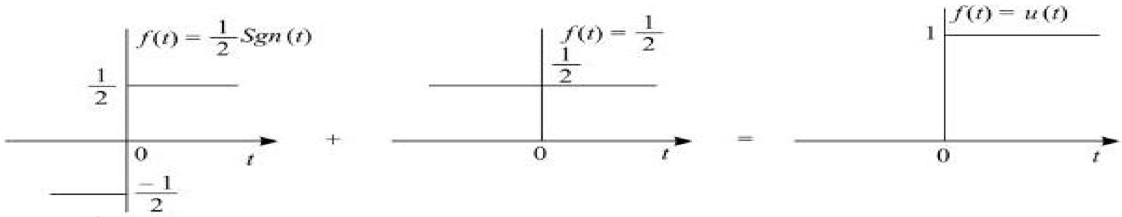
Example 6. Find the Fourier transform of unit step function shown in fig. **(JUNE 2012)(7)**



Solution: The unit step function shown in Fig. is expressed as.

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

The Fourier transform of u(t) can be easily determined using the spectrum of Signum function. The step function can be taken as the sum of Signum and a constant function.



$$u(t) = \frac{1}{2}[1 + Sgn(t)]$$

$$F\{u(t)\} = \frac{1}{2}F[1] + \frac{1}{2}F[Sgn(t)]$$

We know that the Fourier transform of constant is $2\pi\delta(\omega)$

$$F[1] = [F\{A\}]_{A=1} = [2\pi A\delta(\omega)]_{A=1} = 2\pi\delta(\omega)$$

And the Fourier transform of signum function is

$$F[Sgn(t)] = \frac{2}{j\omega}$$

$$F[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$$

Hence

$$F[u(t)] = \pi\delta(\omega) + \frac{1}{j\omega}$$

Property of Fourier Transform: Fourier transform has many important properties. Apart from giving simple solution of complicated Fourier transform. **(DEC 2013)(7)**

1.Linearity Property:

$$f_1(t) \leftrightarrow F_1(\omega)$$
$$f_2(t) \leftrightarrow F_2(\omega)$$
$$f_3(t) \leftrightarrow F_3(\omega)$$
$$f_4(t) \leftrightarrow F_n(\omega)$$

Then

$$a_1 f_1(t) + a_2 f_2(t) + a_3 f_3(t) + \cdots + a_n f_n(t) \leftrightarrow a_1 F_1(\omega) + a_2 F_2(\omega) + \cdots + a_n F_n(\omega)$$

Where $a_1, a_2, a_3, \ldots, a_n$ are the arbitrary constants.

This property is proved easily by linearity property of integrals used in defining Fourier transform.

2.Time scaling Property:

Let $\quad f(t) \leftrightarrow F(\omega)$

Then $\quad f(bt) \leftrightarrow \frac{1}{|b|} F\left(\frac{\omega}{b}\right)$

Where b is real constant.

Proof

$$F[f(bt)] = \int_{-\infty}^{\infty} f(bt)e^{-j\omega t}dt$$

$$bt = x$$

$$dt = \frac{dx}{b}$$

Case 1. when b>0

$$F\{f(bt)\} = \frac{1}{b}\int_{-\infty}^{\infty} f(x)\, e^{-j\left(\frac{\omega}{b}\right)x} dx = \frac{1}{b} F\left(\frac{\omega}{b}\right)$$

Case 2. when b<0

$$f(bt) \leftrightarrow \frac{1}{|b|} F\left(\frac{\omega}{b}\right)$$

Combined the two cases are expressed as,

$$f(bt) \leftrightarrow \frac{1}{|b|} F\left(\frac{\omega}{b}\right)$$

3.Duality Property:

Let	$f(t) \leftrightarrow F(\omega)$
then	$F(t) \leftrightarrow 2\pi f(-\omega)$
or	$F(t) \leftrightarrow f(-f)$

Inverse Fourier transforms of $F(\omega)$ is given by.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t} d\omega$$

t = -t.

Then

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(\omega)\, e^{-j\omega t} d\omega$$

Interchanged the variable t and ω.

$$2\pi f(-\omega) = \int_{-\infty}^{\infty} F(t)\, e^{-j\omega t} dt = F[F(t)]$$

Thus the Fourier transform of the time function F(t) is $2\pi f(-\omega)$

$$F(t) \leftrightarrow 2\pi f(-\omega)$$

For an even function

$$f(-\omega) = f(\omega)$$

Hence

$$F(t) \leftrightarrow 2\pi f(\omega)$$

Example 7. Find the inverse Fourier transform of

(a) $\boldsymbol{Sgn(\omega)}$ **(b)** $\boldsymbol{u(t)}$

Solution (a): The Fourier transform of signum function is.

$$F[\mathrm{sgn}(t)] = \frac{2}{j\omega}$$

$$F\left[\frac{2}{jt}\right] = 2\pi\, Sgn\,(-\omega)$$

by duality property

$Sgn\,(-\omega) = -\,Sgn\,(\omega)$ Because signum is odd function.

Hence

$$\frac{2}{jt} \leftrightarrow -\,2\pi\, Sgn\,(\omega)$$

$$-\frac{1}{jt} \leftrightarrow \pi\, Sgn\,(\omega)$$

$$\frac{j}{\pi t} \leftrightarrow Sgn\,(\omega)$$

Solution (a):

$$u(\omega) = \frac{1}{2} + \frac{1}{2}\, Sgn\,(\omega)$$

$$F^{-1}[u\,(\omega)] = F^{-1}\left[\frac{1}{2}\right] + F^{-1}\left[\frac{1}{2}\, Sgn\,(\omega)\right]$$

by linearity property

$$= \frac{1}{2}\delta\,(t) + \frac{j}{2\pi\, t}$$

4.Time Shifting Property

Let $\qquad f(t) \leftrightarrow F(\omega)$

then

$$f(t-b) \leftrightarrow F(\omega)\, e^{-j\omega b}$$

Proof:

$$f(t-b) \leftrightarrow \int_{-\infty}^{\infty} f(t-b)\, e^{-j\omega t}\, dt$$

Put (t-b) = y so that dt = dy

$$f(t-b) \leftrightarrow \int_{-\infty}^{\infty} f(y)\, e^{-j\omega(b+y)}\, dy = \int_{-\infty}^{\infty} f(y)\, e^{-j\omega y}\, e^{-j\omega b} dy$$

$$f(t-b) \leftrightarrow F(\omega)\, e^{-j\omega b}$$

5.frequency Shifting Property:

If $\qquad f(t) \leftrightarrow F(\omega)$

then

$$e^{j\omega ct} f(t) \leftrightarrow F\,(\omega - \omega_c)$$

Proof: The Fourier transform $F(\omega)$ is given as:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

F(t) = F(t)$e^{j\omega_c t}$

$$f(t)\, e^{j\omega_c t} \leftrightarrow \int_{-\infty}^{\infty} e^{j\omega_c t} f(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} f(t)\, e^{-j(\omega - \omega_c)t}\, dt$$

$$F\,[f(t)\, e^{j\omega_c t}] = F\,(\omega - \omega_c)$$

Example 8. Find the Fourier transform of (a) $cos\ \omega_c\ t$ and (b)) $sin\ \omega_c\ t$ shown inFig. (a) and (c) respectively.

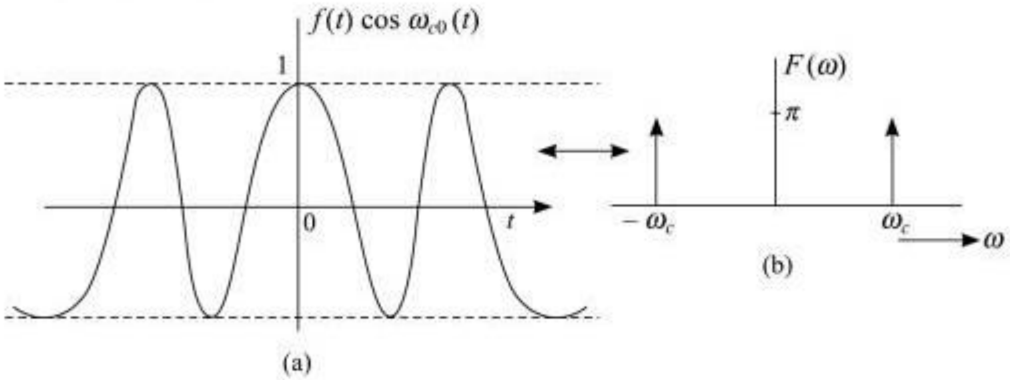
Solution: a)

$$\cos\ \omega_c t = \frac{e^{j\omega_c t} + e^{-j\omega_c t}}{2}$$

$$F\,[\cos\ \omega_c t] = F\left[\frac{1}{2} e^{j\omega_c t}\right] + F\left[\frac{1}{2} e^{-j\omega_c t}\right]$$ by linearity property

The Fourier transform of a constant $\frac{1}{2}$ is $\pi\delta(\omega)$. Therefore, using frequency shifting property.

$$F\,[\cos\ \omega_c t] = \pi\delta\,(\omega - \omega_c) + \pi\delta(\omega + \omega_c)$$



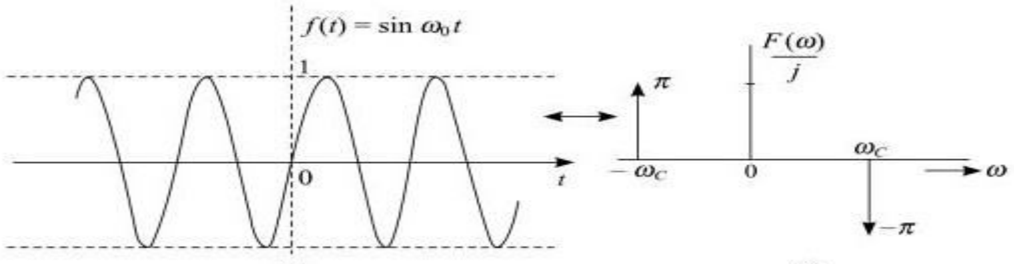
b)

$$\sin\ \omega_c t = \frac{1}{2j} = \left[e^{j\omega_c t} - e^{-j\omega_c t}\right]$$

$$F\,[\sin\ \omega_c t] = F\left[\frac{1}{2j} e^{j\omega_c t}\right] - F\left[\frac{1}{2j} e^{-j\omega_c t}\right]$$ Using Linearity Property

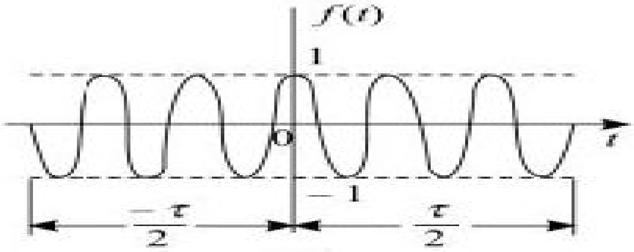
$$F\,[\sin\ \omega_c t] = \frac{1}{j}\,[\pi\delta\,(\omega - \omega_c) - \pi\delta(\omega + \omega_c)]$$ Using Frequency Shifting Property

$$= j\pi\,[\delta\,(\omega + \omega_c) - \delta(\omega - \omega_c)]$$



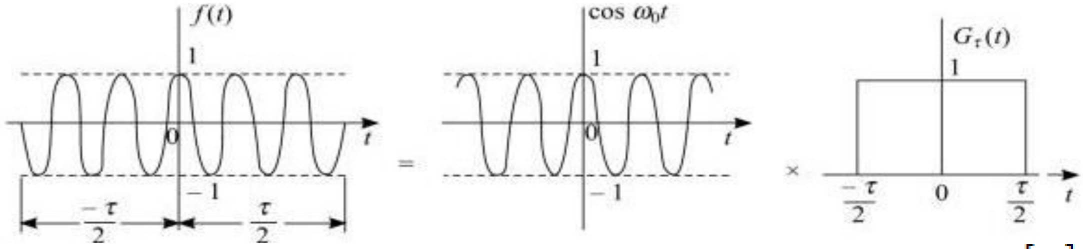
(c) (d)

Example 9. Find the Fourier transform of a radio frequency pulse shown in fig. (a)



(a)

Solution:



The Fourier transform of Gate function is known to be sampling function $k\tau\ Sa\left[\frac{\omega\tau}{2}\right]$ where

$K = 1$

$$F\,[f(t)] = \frac{\tau}{2}\left[Sa\left\{\frac{(\omega-\omega_0)\,\tau}{2}\right\} + Sa\left\{\frac{(\omega+\omega_0)\,\tau}{2}\right\}\right]$$ Using Frequency Shifting Property

	RGPV QUESTIONS	Year	Marks
Q.1	Explain the following properties of Fourier transform: 1. Time Scalingn 2. Duality(Symmetry) 3. Linearity(superposition) 4. Frequency Shifting	DEC 2013 JUNE 2012 JUNE 2010	7 7 10
Q.2	Calculate the fourier transform of a given function along with spectrum analysis: f(t)= u(t) where u(t) is a unit step function.	JUNE 2012 DEC 2010	7 10

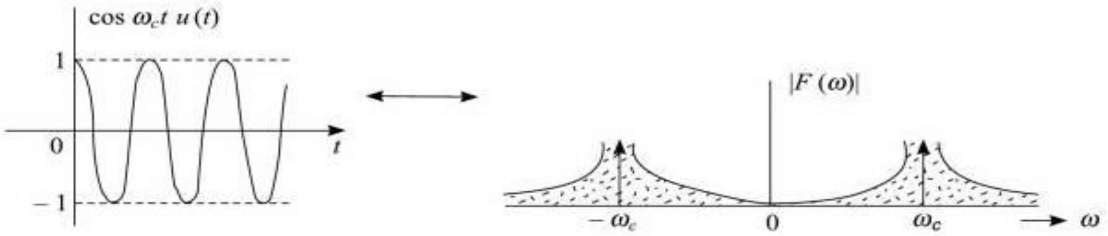
UNIT-01/LECTURE-04

NUMERICAL AND PROPERTY OF FOURIER TRANSFORM

Example 10: An audio oscillator is switched on at t = 0. Find the spectrum of the generated (a) cosine waveform and (b) sine waveform, both of frequency ω_c. **(DEC 2011)(10)**

Solution: (a)

$$u(t)\ \cos\ \omega_c t\ =\ \frac{1}{2}\,u(t)\left[e^{j\omega_c t} + e^{-j\omega_c t}\right]$$



$$F[u(t)\ \cos\omega_c\ t]\ =\ F\left[\frac{1}{2}u(t)\,e^{j\omega_c t} + \frac{1}{2}u\ (t)\,e^{-j\omega_c t}\right]$$

We know that the Fourier transform of unit step function is

$$u(t) \leftrightarrow \pi\delta(\omega) + \frac{1}{j\omega}$$

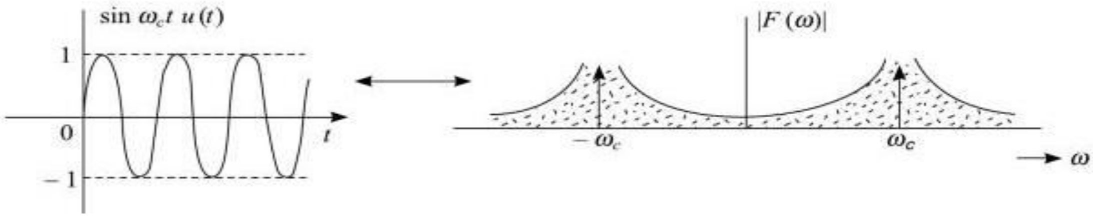
$$F[u(t)\ \cos\omega_c\ t]\ =\ \frac{\pi}{2}\delta(\omega-\omega_c) + \frac{1}{2j(\omega-\omega_c)} + \frac{\pi}{2}\delta(\omega+\omega_c) + \frac{1}{2j(\omega+\omega_c)}$$ by Frequency Shifting Property

$$=\ \frac{\pi}{2}[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)] + \frac{1}{2j(\omega-\omega_c)} + \frac{1}{2j(\omega+\omega_c)}$$

$$u(t)\cos\omega_c\ t\ =\ \frac{\pi}{2}[\delta(\omega-\omega_c)+\delta(\omega+\omega_c)] + \frac{j\omega}{\omega_c^2-\omega^2}$$

(a)

$$u(t)\ \sin\omega_c\ t\ =\ \frac{1}{2j}\,u(t)\left[e^{j\omega_c t} - e^{-j\omega_c t}\right]$$

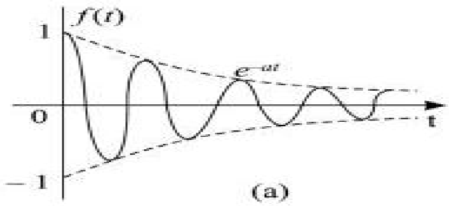


$$u(t)\sin\omega_c\ t \leftrightarrow \frac{\pi}{2j}[\delta(\omega-\omega_c)-\delta(\omega+\omega_c)] + \frac{\omega_c}{\omega_c^2-\omega^2}$$

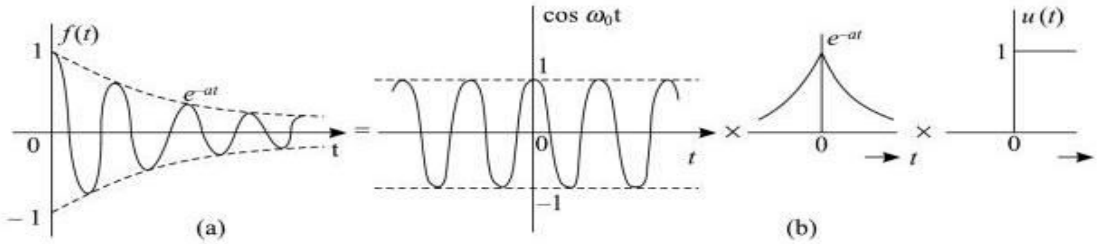
$$F[u(t)\sin\omega_c t)] \leftrightarrow F\left[\frac{1}{2j}u(t)e^{j\omega_c t} - \frac{1}{2j}u(t)e^{-j\omega_c t}\right]$$

by Frequency Shifting Property

Example 11: Find the Fourier transform of a damped sinusoidal waveform of frequency ω_0 shown in fig. (a) **(DEC 2011)(10)**



Solution:



$$f(t) = e^{-at}\cos\omega_0 t\, u(t)$$

We know that the Fourier transform of

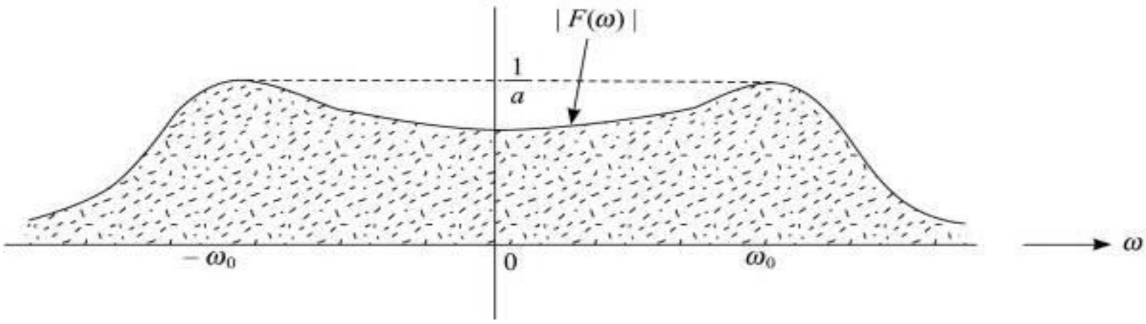
$$e^{-at}u(t) \leftrightarrow \frac{1}{a+j\omega}$$

$$F\left[e^{-at}\cos\,\omega_0 t\, u(t)\right] = F\left[\frac{1}{2}e^{-at}\,u(t)\,e^{j\omega_0 t} + \frac{1}{2}e^{-at}\,u(t)\,e^{-j\omega_0 t}\right]$$

$$F(\omega) = \frac{1}{2}\left[\frac{1}{a+j(\omega-\omega_0)} + \frac{1}{a-j(\omega-\omega_0)}\right]$$ Using Frequency Shifting Property

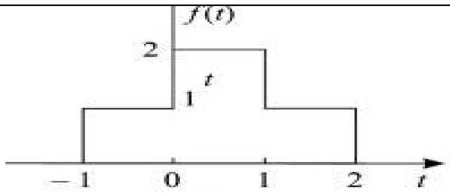
$$= \frac{a}{a^2+(\omega-\omega_0)^2}$$

$$|F(\omega)|_{\omega=\omega_0} = \frac{1}{a}$$

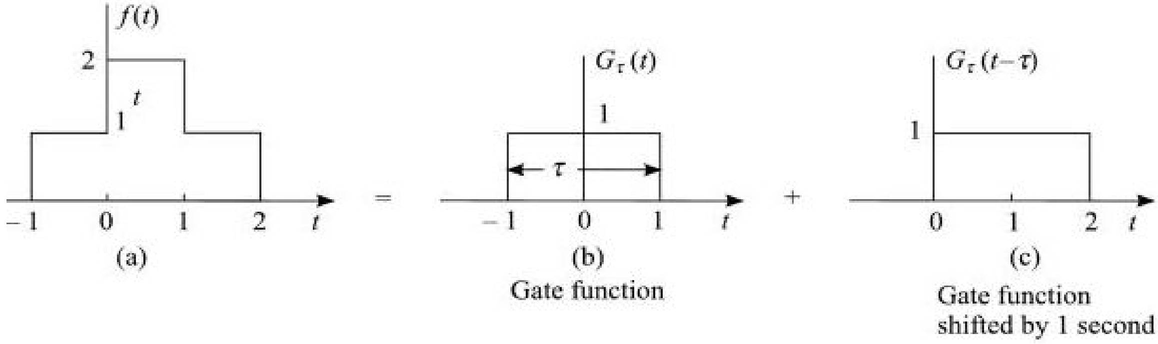


Spectrum of given damped sinusoidal waveform

Example 12: Find the Fourier transform of the waveform shown in fig.



Solution: The waveform representing f(t) may be expressed as the sum of two waveform shown in fig.



We know that the Fourier transform of Gate function is $k\tau Sa\left[\frac{\omega\tau}{2}\right]$ where k = 1 and $\tau = 2$.

Fourier transform of fig (b) is $2Sa[\omega]$

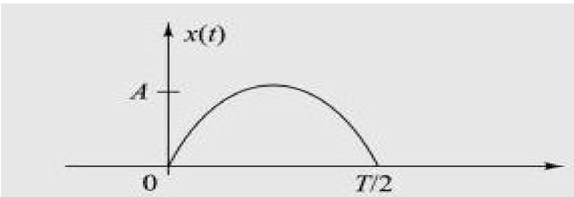
Fourier transform of fig (c) is $2Sa[\omega - 1]$ Using Frequency Shifting Property

The Fourier transform of given function is $2Sa[\omega] + 2Sa[\omega - 1]$ Using Linearity Property

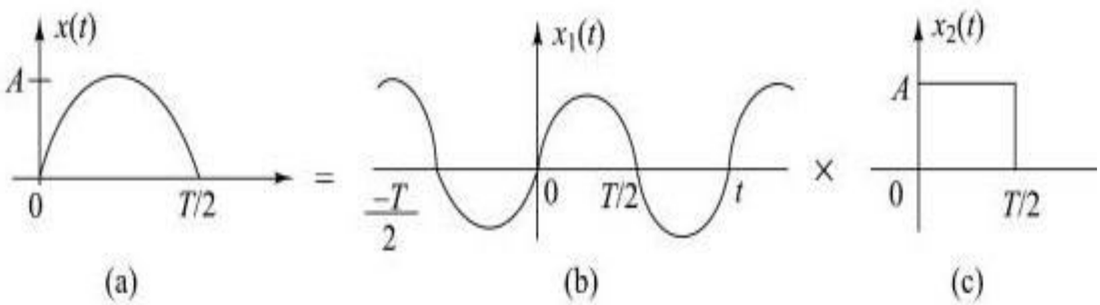
$$F[f(t)] = 2Sa(\omega) + 2Sa(\omega)e^{-j\omega}$$

$$f(t) \leftrightarrow 2Sa(\omega)[1 + e^{-j\omega}]$$

Example 13: Find the Fourier transform of the half sinusoid waveform shown in fig.



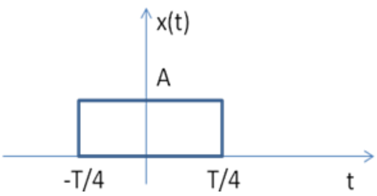
Solution: The waveform representing x(t) may be expressed as the product of two waveform shown in fig (b) and (c)



$$x(t) = x_1(t) \times x_2(t)$$
$$= \sin \omega_0 t \times x_2(t)$$
$$= \left[\frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j}\right] \times x_2(t)$$
$$= \frac{1}{2j} e^{j\omega_0 t} x_2(t) - \frac{1}{2j} e^{-j\omega_0 t} x_2(t)$$

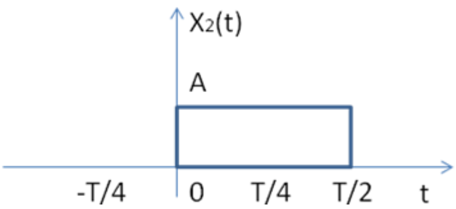
The Fourier transform of x_2(t) can be obtained using FT of Gate function and then applying

time shofting property.
We know that the FT of Gate function is $k\tau Sa\left[\frac{\omega\tau}{2}\right]$



k=A and $\tau = T/2$

So that the FT of this function is $\frac{AT}{2} Sa\left[\frac{\omega T}{4}\right]$



Fourier transform of $x_2(t) = \frac{AT}{2} Sa\left[\frac{\omega T}{4}\right] e^{\frac{-j\omega T}{4}}$

So

$$F[x(t)] = F\left[\frac{1}{2j} e^{j\omega_0 t} x_2(t)\right] - F\left[\frac{1}{2j} e^{-j\omega_0 t} x_2(t)\right]$$
$$= \frac{AT}{4j} Sa\left[(\omega+\omega_0)\frac{T}{4}\right] e^{-j(\omega+\omega_0)\frac{T}{4}} - \frac{AT}{4} Sa\left[(\omega-\omega_0)\frac{T}{4}\right] e^{-j(\omega-\omega_0)\frac{T}{4}}$$

Using Time Shifting Property

6.Time Differentoation and Integration Property:

$$\frac{df(t)}{dt} \leftrightarrow j\omega F(\omega)$$

Proof:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega$$

$$\frac{df(t)}{dt} = \frac{1}{2\pi}\frac{d}{dt}\left[\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}d\omega\right]$$

$$\frac{df(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d}{dt}\{F(\omega)\,e^{j\omega t}\}\,d\omega$$

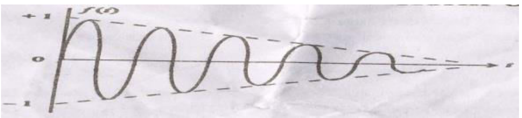
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega\,F(\omega)\,e^{j\omega t}\,d\omega = F^{-1}[\,j\omega F(\omega)]$$

$$F\left[\frac{d\,f(t)}{dt}\right] = j\omega\,F(\omega)$$

$$\frac{d\,f(t)}{dt} \leftrightarrow j\omega\,F(\omega)$$

$$\frac{d^n f(t)}{dt^n} = (j\omega)^n F(\omega)$$

In general

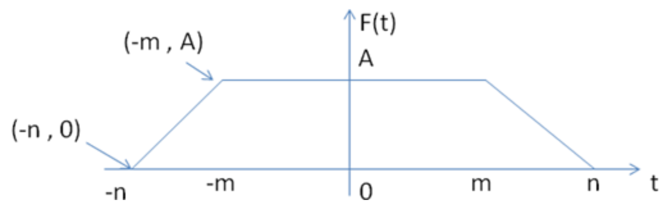
	RGPV QUESTIONS	Year	Marks
Q.1	An audio oscillator is switched on at t = 0, find the spectrum of the generated cosine waveform and sine waveform, both of frequency wc	DEC 2011	10
Q.2	Determine the Fourier transform of a damped sinusoidal waveform of frequency wo as shown in figure. 	DEC 2011	10

UNIT-01/LECTURE-05

FOURIER TRANSFORM OF PERIODIC FUNCTIONS

Example 14: Determine the Fourier transform of a trapezoidal function F(t) shown in Fig.

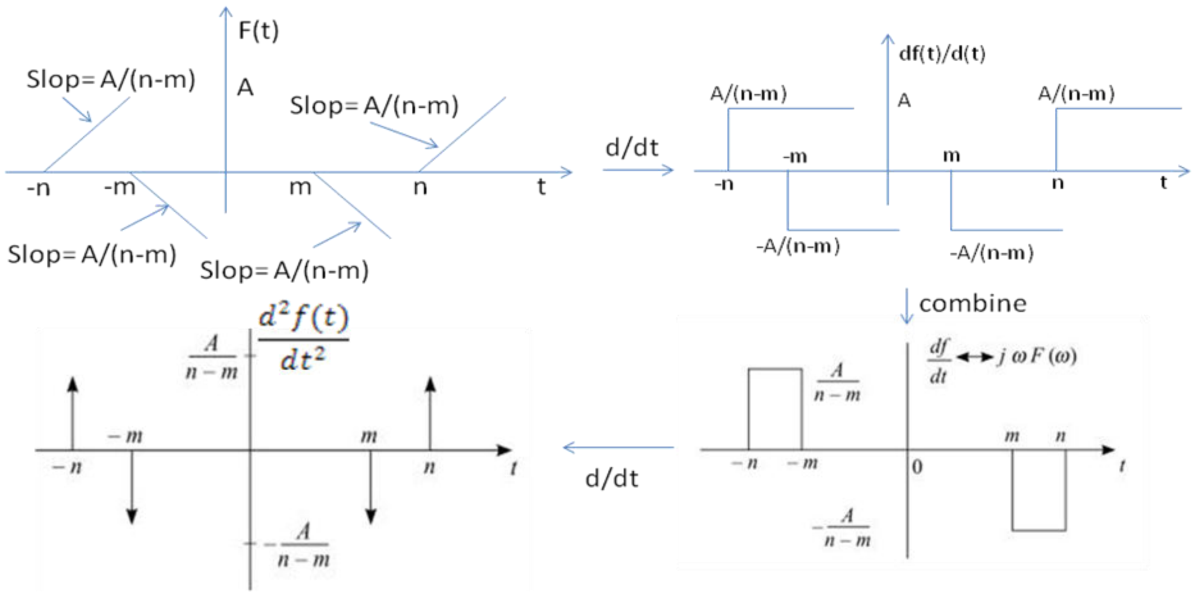
(DEC 2010)(7)



A Trapezoidal Function

Solution: give function made by ramp function $slop = \frac{y_2 - y_1}{x_2 - x_1} = \frac{A-0}{-m-(-n)} = \frac{A}{n-m}$

Derivative of ramp function is uhit step function and Derivative of step is impulse.



$$\frac{d^2 f(t)}{dt^2} = \frac{A}{n-m}\delta(t+n) - \frac{A}{n-m}\delta(t+m) - \frac{A}{n-m}\delta(t-m) + \frac{A}{n-m}\delta(t-n)$$

$$\frac{d^2f(t)}{dt^2} = \frac{A}{n-m}[\delta(t+n) - \delta(t+m) - \delta(t-m) + \delta(t-n)]$$

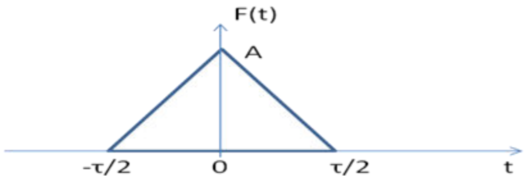
Take Fourier transform of above equation in both the side
We know that the Fourier transform of impulse function is $\delta(t) = 1$

and $\delta(t+n) = 1 * e^{j\omega n}$ Using Time Shifting Property

$$(j\omega)^2 F(\omega) = \frac{A}{n-m}\left[1 * e^{j\omega n} - 1 * e^{j\omega m} - 1 * e^{-j\omega m} + 1 * e^{-j\omega n}\right]$$

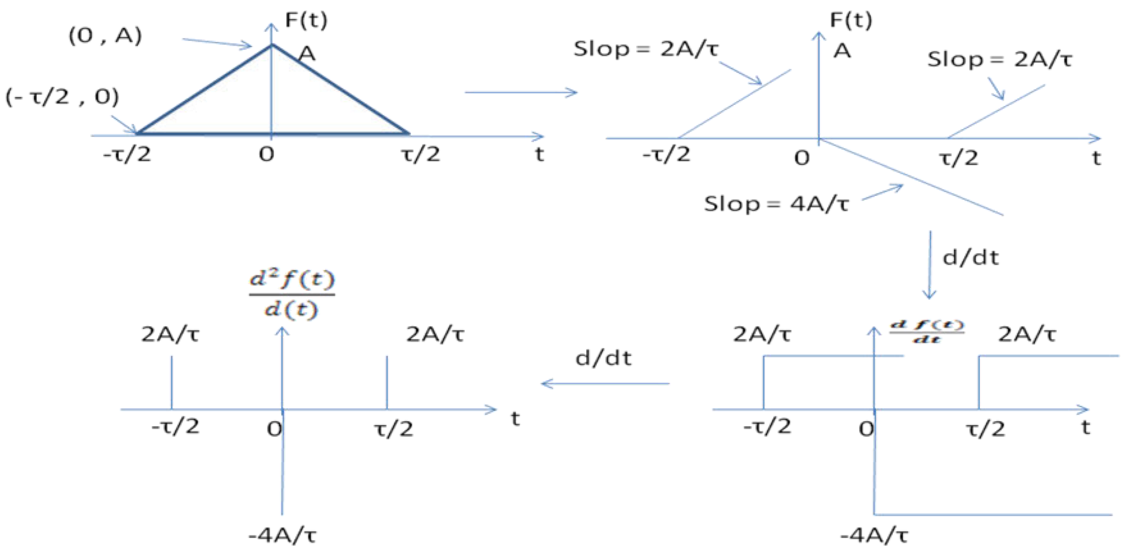
$$F(\omega) = \frac{2A}{(n-m)}\left(\frac{\cos m\omega - \cos n\omega}{\omega^2}\right)$$

Example 15: Determine the Fourier transform of a triangular pulse shown in Fig.



Solution: $slop = \frac{y_2 - y_1}{x_2 - x_1} = \frac{A-0}{0-\frac{-\tau}{2}} = \frac{2A}{\tau}$

Derivative of ramp function is uhit step function and Derivative of step is impulse.



$$\frac{d^2f(t)}{dt^2} = \frac{2A}{\tau}\delta\left(t+\frac{\tau}{2}\right) - \frac{4A}{\tau}\delta(t) + \frac{2A}{\tau}\delta\left(t-\frac{\tau}{2}\right)$$

Take Fourier transform of above equation in both the side
We know that the Fourier transform of impulse function is $\delta(t) = 1$

and $\delta\left(t+\frac{\tau}{2}\right) = 1 * e^{j\omega\frac{\tau}{2}}$ Using Time Shifting Property.

$$(j\omega)^2F(\omega) = \frac{2A}{\tau}\left[1 * e^{j\omega\frac{\tau}{2}} - 2 * 1 + 1 * e^{-j\omega\frac{\tau}{2}}\right]$$

$$\omega) = \frac{2A}{\tau}\left\{2\left[\frac{e^{j\omega\frac{\tau}{2}} + e^{-j\omega\frac{\tau}{2}}}{2}\right] - 2\right\}$$

$$(j\omega)^2F(\omega) = \frac{2A}{\tau}\left\{2\cos\omega\frac{\tau}{2} - 2\right\}$$

$$-\omega^2F(\omega) = \frac{4A}{\tau}\left\{\cos\omega\frac{\tau}{2} - 1\right\}$$

$$\omega^2F(\omega) = \frac{4A}{\tau}\left\{1 - \cos\omega\frac{\tau}{2}\right\} \qquad \cos 2\theta = 1 - 2\,sin^2\,\theta$$

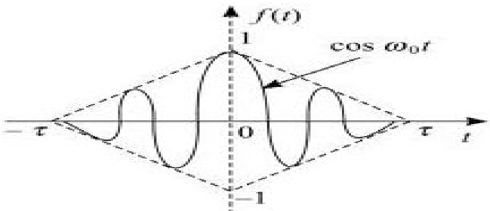
$$2\,sin^2\,\frac{\theta}{2} = 1 - \cos\theta$$

$$\omega^2F(\omega) = \frac{4A}{\tau}\,2\,sin^2\,\frac{\omega\tau}{4}$$

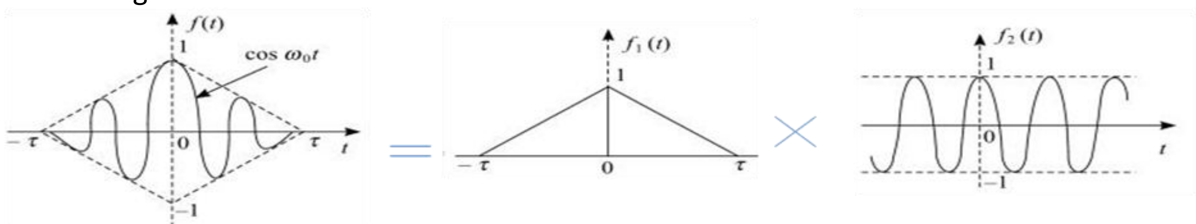
$$F(\omega) = \frac{\frac{8A}{\tau}sin^2\frac{\omega\tau}{4}}{\omega^2} = \frac{\frac{8A}{\tau}sin^2\frac{\omega\tau}{4}}{\omega^2\frac{\tau^2}{16}} * \frac{\tau^2}{16} = \frac{sin^2\frac{\omega\tau}{4}}{\left(\frac{\omega\tau}{4}\right)^2} * \frac{\tau^2}{16} * \frac{8A}{\tau} = \frac{A\tau}{2}Sa^2\left(\frac{\omega\tau}{4}\right)$$

$$Sa(x) = \frac{Sin\,x}{x} \text{ and } Sa^2(x) = \frac{Sin^2(x)}{x^2}$$

Example 16: Determine the Fourier transform of a triangular RF pulse shown in Fig. and draw the spectrum. The radio frequency is ω_0 .



Solution: The waveform representing f(t) may be expressed as the product of two waveform shown in fig.



$$f(t) = f_1(t) * f_2(t) = f_1(t) * Cos\omega_0 t = f_1(t) * \frac{[e^{j\omega t} + e^{-j\omega t}]}{2} = \frac{1}{2}[f_1(t) * e^{j\omega t} + f_1(t) * e^{-j\omega t}]$$

$f_1(t)$ *is a triangular Pulse*

We know that the Fourier transform of triangular pulse is $= \frac{A\tau}{2} Sa^2 \left(\frac{\omega\tau}{4}\right)$

$f_1(t) = \tau\, Sa^2\left(\frac{\omega\tau}{2}\right)$ *where* $A = 1$ *and* $\tau = 2\tau$ *(A is amplitude and* τ *is duration of signal)*

$$f(t) = \frac{1}{2}\left[\tau\, Sa^2\,(\omega - \omega_0)\frac{\tau}{2}\right] + \left[\tau\, Sa^2\,(\omega + \omega_0)\frac{\tau}{2}\right]$$

$$f(t) = \frac{\tau}{2} Sa^2\left[(\omega - \omega_0)\frac{\tau}{2}\right] + \frac{\tau}{2} Sa^2\left[(\omega + \omega_0)\frac{\tau}{2}\right]$$

Fourier Transform of Periodic Functions

However once we determine the Fourier transform of periodic function the Fourier transform will provide as a unified tool for analyzing both periodic and non-periodic waveform over the entire interval. this can be done using the concept of Delta function. Fourier transform of a periodic function can be determine in limited cases, as was done for sinusoidal function, in spite of the fact that periodic function fails to satisfy the condition of absolute integrability.

Let us now find Fourier transform of periodic function f(t) with time period T. The function can be expressed in terms of complex Fourier series.

$$f(t) = \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t},\ \omega_0 = \frac{2\pi}{T}$$

Taking Fourier transform of both the side

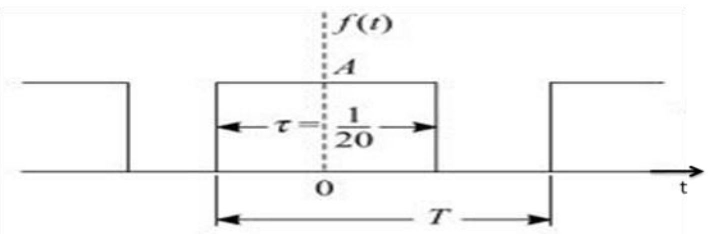
$$F[f(t)] = F\left[\sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}\right] = \sum_{n=-\infty}^{\infty} F_n\, F\,[1\cdot e^{jn\omega_0 t}]$$

$$F[1\cdot e^{jn\omega_0 t}] = 2\pi\delta\,(\omega - n\omega_0)$$ Using Frequency Shifting Property

$$F[f(t)] = 2\pi \sum_{n=-\infty}^{\infty} F_n\ \delta(\omega - n\omega_0)$$

Where $F_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_1(t) e^{-jn\omega_0 t}\, dt$

Example17: Find the Fourier transform of a periodic Gate function with $T = \frac{1}{2}$ and width $\tau = \frac{1}{20}$

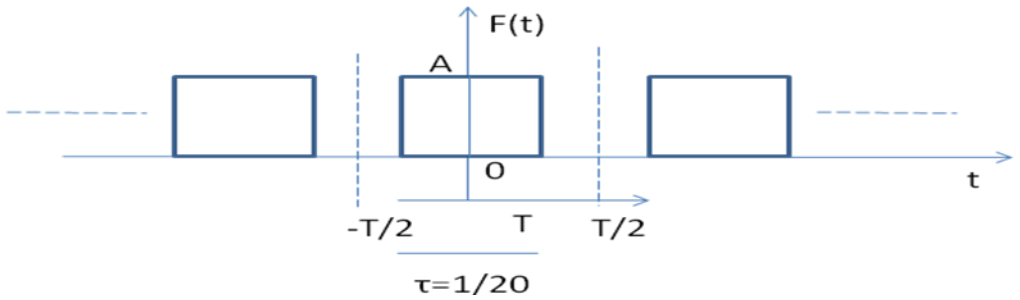


Solution:

$F[f(t)] = 2\pi \sum_{n=-\infty}^{\infty} F_n \ \delta(\omega - n\omega_0)$ ....(1)

The value of Fn is $F_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f_1(t) e^{-jn\omega_0 t}\, dt$

Function f(t) has only one Gate pulse in interval $\left(-\frac{T}{2}, \frac{T}{2}\right)$. The Fn can be written as



So $F_n = \frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} f_1(t) e^{-jn\omega_0 t}\, dt = \frac{1}{T}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} A\, e^{-jn\omega_0 t}\, dt = \frac{A}{T}\left[\frac{e^{-jn\omega_0 t}}{-jn\omega_0}\right]_{\frac{-\tau}{2}}^{\frac{\tau}{2}}$

$$F_n = \frac{A}{-jn\omega_0 T}\left[e^{\frac{-jn\omega_0\tau}{2}} - e^{\frac{jn\omega_0\tau}{2}}\right] = \frac{2*A}{n\omega_0 T}\left[\frac{e^{\frac{jn\omega_0\tau}{2}} - e^{\frac{-jn\omega_0\tau}{2}}}{2j}\right] = \frac{2*A}{n\omega_0 T}\left[\sin n\omega_0 \frac{\tau}{2}\right]$$

$$F_n = \frac{2*A}{T}\frac{\left[\sin n\omega_0 \frac{\tau}{2}\right]}{n\omega_0} = \frac{2*A}{T} * \frac{\tau}{2}\frac{\left[\sin n\omega_0 \frac{\tau}{2}\right]}{n\omega_0 \frac{\tau}{2}} = \frac{A\tau}{T} Sa(n\omega_0\tau/2) = \frac{A\tau}{T} Sa(n\pi\tau/T)$$

Given data width $\tau = \frac{1}{20}$ $and\ T = \frac{1}{2}$ $then$ $\frac{\tau}{T} = \frac{1/20}{1/2} = \frac{2}{20} = \frac{1}{10}$

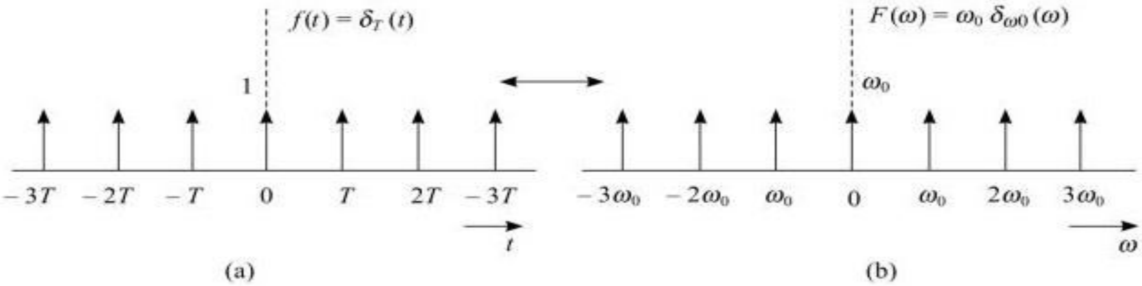
$$F_n = \frac{A}{10} Sa(n\pi/10)$$

the Fourier transform is obtained by substituting this value of Fn in equation no (1).

$$F[f(t)] = 2\pi \sum_{n=-\infty}^{\infty} \frac{A}{10} Sa\left(\frac{n\pi}{10}\right) \delta[\omega - n\omega_0]$$

$$F(\omega) = \frac{2\pi A}{10} \sum_{n=-\infty}^{\infty} Sa\left(\frac{n\pi}{10}\right) \delta[\omega - n\omega_0]$$

Example18: Prove that a Dirac comb is its own Fourier transform. **(JUNE 2013)(7)**
Solution: A Dirac comb is a comb-like waveform consisting of a sequency of equidistant impulses shown in Fig.



The function may be expressed as

$\delta_T(t) = \delta(t) + \delta(t-T) + \delta(t-2T) \dots . \delta(t-nT) \dots . \delta(t+T) + \delta(t+2T) \dots . \delta(t+nT) \dots$

$$F[f(t)] = 2\pi \sum_{n=-\infty}^{\infty} F_n \; \delta(\omega - n\omega_0)$$

The Fourier transform of the Dirac comb may be obtained by above equation. The value of F_n is evaluated by

$$F_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} f_1(t) e^{-jn\omega_0 t} \, dt$$

Function $\delta_T(t)$ has only one impulse in the interval $\left[\frac{-T}{2}, \frac{T}{2}\right]$.

$$F_n = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-jn\omega_0 t} \, dt$$

Integration of impulse function for same limit is the value of function at t=0.(Using Sampling Property of unit impulse function)

$$F_n = \frac{1}{T} \boxed{\int_{-\frac{T}{2}}^{\frac{T}{2}} \delta(t) e^{-jn\omega_0 t} \, dt}$$ t=0

$$F_n = \frac{1}{T} e^{-jn\omega_0 * 0} = \frac{1}{T}$$

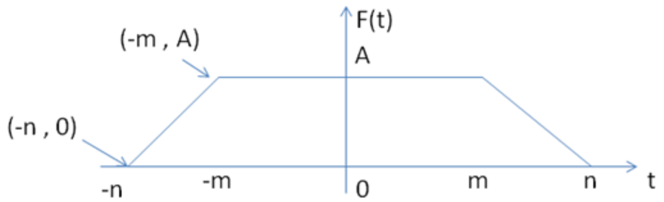
The Fourier transform of given function is

$$F[\delta_T(t)] = 2\pi \sum_{n=-\infty}^{\infty} F_n \; \delta(\omega - n\omega_0) = 2\pi \sum_{n=-\infty}^{\infty} \frac{1}{T} \delta(\omega - n\omega_0)$$

$$= \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

Where $T = \frac{2\pi}{\omega_0}$

$$\delta_T(t) \leftrightarrow \omega_0 \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_0)$$

	RGPV QUESTIONS	Year	Marks
Q.2	Show that the Fourier transform of a diac comb is a diac comb itself.	JUNE 2013 DEC 2012	7 7
Q.1	Evaluate the Fourier transform of a trapezoidal function shown in fig. 	DEC 2010	10

UNIT-01/LECTURE-06

CONCEPT OF ENERGY DENSITY (PARSEVAL'S THEOREM)

A primary goal of the communication system is to transmit more signal power as against noise power to achieve greater signal to noise ratio. Hence, for evaluation of signal to noise power ratio, it is necessary to evolve a method for calculating the power content of a signal.

Energy signal:- The energy of a signal exists only if the integral in equation is finite.

$$E = \int_{-\infty}^{\infty} f^2(t)dt$$

The signals for which equation is finite are called energy signals. Aperiodic signals are examples of energy signals.

Parseval's Theorem for Energy Signals:- **(JUNE 2013)(7)**

The Parseval's theorem defines energy of a signal in terms of its Fourier transform. The theorem is very useful as it helps in evaluating the energy of a signal without knowing its time domain. When the Fourier transform of a signal is known, its energy can be evaluated without finding the inverse Fourier transform i.e., Fourier transform of f(t) if $F(\omega)$. The energy E of f(t) is given by

$$E = \int_{-\infty}^{\infty} f^2(t)\, dt = \int_{-\infty}^{\infty} f(t).f(t)\, dt$$

Replacing second f(t) in terms of the inverse Fourier transform of $F(\omega)$, we gate

$$E = \int_{-\infty}^{\infty} f(t) \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} d\omega \right\} dt$$

By interchanging the order of integration,

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \left\{ \int_{-\infty}^{\infty} f(t)\, e^{j\omega t} dt \right\} d\omega$$

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) F(-\omega)\, d\omega$$

$$\int_{-\infty}^{\infty} f(t) e^{j\omega t} dt = F(-\omega)$$

For a real function f(t), the Fourier transform $F(\omega)$ and $F(-\omega)$ are complex conjugates.

$$F(\omega)F(-\omega) = |F(\omega)|^2$$

Hence

$$E = \int_{-\infty}^{\infty} f^2(t)\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

$$E = \frac{1}{2\pi} [Area\ under\ |F(\omega)|^2\ curve]$$

Above equation is called Parseval,s theorem for energy signals.

Example17: A signal $e^{-3t}u(t)$ is passed through an ideal low pass filter with cut off frequency of 1 rad per second. **(DEC 2013)(14)**

(a) Test whether the input is an energy signal.
(b) Find the input output energy.

Solution: The input signal f(t) is given by

$$f(t) = e^{-3t}u(t)$$

The energy of the input signal is given as

$$Ei = \int_{-\infty}^{\infty} f^2\,(t)\,dt$$

Since unit step vanishes for t < 0, hence t=0 is taken as lower limit of the integral

$$Ei = \int_0^{\infty} (e^{-3t})^2\,dt$$

$$Ei = \int_0^{\infty} e^{-6t}\;\;dt$$

$$Ei = \frac{e^{-6t}}{-6} \quad \text{for lower limit 0 and upper limit } \infty$$

$$Ei = \frac{1}{6}$$

Since the energy is finite, the input signal is an energy signal.

(a) The energy density spectrum of the output r(t) is given as

$$\Psi_r\,(w) = \Psi_f\,(w) \quad |H(w)|^2$$

Here

$$\Psi_f(w) \;=\; |F(w)|^2$$

$$F(w) \;= F[\,e^{-3t}\,\mathrm{u(t)}]$$

$$F(w) \;=\; \frac{1}{jw+3}$$

Hence

$$\Psi_f(w) \;=\; |F(w)|^2 = \left|\frac{1}{jw+3}\right|^2$$

$$\Psi_f(w) \;=\; \left|\frac{1}{jw+3} \times \frac{jw-3}{jw-3}\right|^2$$

$$\Psi_f(w) \;=\; \left|\frac{3-jw}{w^2+9}\right|^2$$

$$\Psi_f(w) \;=\; \frac{3^2+w^2}{(w^2+9)\times(w^2+9)}$$

$$\Psi_f(w) \;=\; \frac{w^2+\;9}{(w^2+9)\;\times\;(w^2+9)}$$

$$\Psi_f(w) \;=\; \frac{1}{w^2+9}$$

The square of the transfer function of the low pass filter is given by

$$|H(w)|^2 = 1 \qquad for\ |w| < 1$$
$$|H(w)|^2 = 0 \qquad otherwise$$

Which gives

$$\Psi_r(w) = \Psi_f(w) \quad |H(w)|^2$$

$$\Psi_r(w) = \frac{1}{w^2+9} \times \; 1 \qquad for\; |w| < 1$$

$$\Psi_r(w) = \frac{1}{w^2+9}$$

Total energy contained in output is given by

$$Eo \; = \; \frac{1}{\pi}\int_0^{\infty} \Psi_r(w)\; dw$$

$$Eo \; = \; \frac{1}{\pi}\int_0^{\infty} \frac{1}{w^2+9}\; dw$$

$$Eo \; = \; \frac{1}{3\pi}\; tan^{-1}\left(\frac{1}{3}\right)$$

Energy Spectral Density:-

Energy spectral density describes how the energy of a signal or a time series is distributed with frequency. Here, the term energy is used in the generalized sense of signal processing that is, the energy of a signal x(t) is

$$\int_{-\infty}^{\infty} |x(t)|^2 dt.$$

The energy spectral density is most suitable for transients—that is, pulse-like signals—having a finite total energy. In this case, Parseval's theorem gives us an alternate expression for the

$$\hat{x}(f) = \int_{-\infty}^{\infty} e^{-2\pi i f t} x(t) dt.$$

energy of the signal in terms of its Fourier transform.

$$\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} |\hat{x}(f)|^2 \, df.$$

Here f is the frequency in Hz, i.e., cycles per second. Often used is the angular frequency $\omega = 2\pi f$. Since the integral on the right-hand side is the energy of the signal, the integrand $|\hat{x}(f)|^2$ can be interpreted as a density function describing the energy per unit frequency contained in the signal at the frequency f. In light of this, the energy spectral density of a signal x(t) is defined as

$$S_{xx}(f) = |\hat{x}(f)|^2$$

As a physical example of how one might measure the energy spectral density of a signal, suppose V(t) represents the potential (in volts) of an electrical pulse propagating along a transmission line of impedance Z, and suppose the line is terminated with a matched resistor (so that all of the pulse energy is delivered to the resistor and none is reflected back). By Ohm's law, the power delivered to the resistor at time t is equal to $V(t)^2/Z$, so the total energy is found by integrating $V(t)^2/Z$ with respect to time over the duration of the pulse. To find the value of the energy spectral density $S_{xx}(f)$ at frequency f, one could insert between the transmission line and the resistor a band pass filter which passes only a narrow range of frequencies (Δf, say) near the frequency of interest and then measure the total energy $E(f)$ dissipated across the resistor. The value of the energy spectral density at f is then estimated to be $E(f)/\Delta f$. In this example, since the power $V(t)^2/Z$ has units of $V^2\ \Omega^{-1}$, the energy $E(f)$ has units of $V^2\ s\ \Omega^{-1}$ = J, and hence the estimate $E(f)/\Delta f$ of the energy spectral density has units of J Hz^{-1}, as required. In many situations, it is common to forgo the step of dividing by Z so that the energy spectral density instead has units of $V^2\ s\ Hz^{-1}$.

This definition generalizes in a straightforward manner to a discrete signal with an infinite number of values x_n such as a signal sampled at discrete times $x_n = x(n\,\Delta t)$:

$$S_{xx}(f) = (\Delta t)^2 \left| \sum_{n=-\infty}^{\infty} x_n e^{-2\pi i f n} \right|^2 = (\Delta t)^2 \hat{x}_d(f) \hat{x}_d^*(f),$$

Where $\hat{x}_d(f)$ is the discrete Fourier transform of x_n. The sampling interval Δt is needed to keep the correct physical units and to ensure that we recover the continuous case in the limit $\Delta t \to 0$; however in the mathematical sciences, the interval is often set to 1.

	RGPV QUESTIONS	Year	Marks
Q.1	A signal $e^{-3t}u(t)$ is passed through an ideal low pass filter with cut off frequency of 1 rad per second. (a) Whether the input is an energy signal. (b) Find the input output energy.	DEC 2013 JUNE 2012 JUNE 2011 DEC 2010	14 7 10 10
Q.2	What is energy signal? State & prove Parseval's theorem for energy signals.	JUNE 2013	7
Q.3	State & prove Parseval's theorem for energy signals.	JUNE 2012 DEC 2011 DEC 2010	7 10 10
Q.5	Explain parseval's theorem for energy signals.	JUNE 2011	10

UNIT-01/LECTURE-07

CONCEPT OF POWER DENSITY (PARSEVAL'S THEOREM)

Power signal:- Signals having infinity energy, but finite average power, are called power signals.

Parseval's Theorem for Power Signals:- **(JUNE 2013)(7)**

This theorem is similar to Parseval's theorem for energy signals. The theorem defined the power of a signal in terms of its Fourier series coefficients, i.e., in terms of amplitudes of the harmonic components present in the signal.

We know that $|f(t)|^2 = f(t)f^*(t)$

Where $f^*(t)$ is the complex conjugate of the $f(t)$. The power of the signal $f(t)$ over a cycle is given by,

$$P = \frac{1}{T}\int_{-T/2}^{T/2} |f(t)|^2\, dt = \frac{1}{T}\int_{-T/2}^{T/2} f(t)f^*(t)\, dt$$

Replacing $f(t)$ by its exponential Fourier series,

$$P = \frac{1}{T}\int_{-T/2}^{T/2} f^*(t) \sum_{n=-\infty}^{\infty} F_n e^{jn\omega_0 t}\, dt$$

Where $\omega_0 = \frac{2\pi}{T}$

Interchanging the order of integration and summation,

$$P = \frac{1}{T}\sum_{n=-\infty}^{\infty} F_n \int_{-T/2}^{T/2} f^*(t) e^{jn\omega_0 t}\, dt$$

The integral in the above expression is equal to $T\ F_n$. Hence we may write

$$P = \frac{1}{T}\sum_{n=-\infty}^{\infty} F_n F_n^* = \frac{1}{T}\sum_{n=-\infty}^{\infty} |F_n|^2$$

Above equation is known as Parseval,s Power Theorem. The equation defines that the power of the signal is to the sum of the square of the magnitudes of various harmonics present in discrete spectrum. This is a special case of Parseval's theorem defined earlier for energy signals having continuous spectrum. $|F_n|^2$ is referred as the discrete power spectrum of the signal $f(t)$.

Power Spectral Density:-

The above definition of energy spectral density is most suitable for transients, i.e., pulse-like signals, for which the Fourier transforms of the signals exist. For continued signals that describe, for example, stationary physical processes, it makes more sense to define a power spectral density (PSD), which describes how the power of a signal or time series is distributed over the different frequencies, as in the simple example given previously. Here, power can be the actual physical power, or more often, for convenience with abstract signals, can be defined as the squared value of the signal. For example, statisticians study the variance of a set of data, but because of the analogy with electrical signals, it is customary to refer to it as the power spectrum even when it is not, physically speaking, power. The average power P of a signal x(t) is the following time average:

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)^2 \, dt.$$

The power of a signal may be finite even if the energy is infinite. For example, a 10-volt power supply connected to a 1 kΩ resistor delivers $(10\ \mathrm{V})^2 / (1\ \mathrm{k}\Omega) = 0.1\ \mathrm{W}$ of power at any given time; however, if the supply is allowed to operate for an infinite amount of time, it will deliver an infinite amount of energy (0.1 J each second for an infinite number of seconds).

In analyzing the frequency content of the signal $x(t)$, one might like to compute the ordinary Fourier transform $\hat{x}(\omega)$; however, for many signals of interest this Fourier transform does not exist.[N 1] Because of this, it is advantageous to work with a truncated Fourier transform $\hat{x}_T(\omega)$, where the signal is integrated only over a finite interval [0 , T]:

$$\hat{x}_T(\omega) = \frac{1}{\sqrt{T}} \int_0^T x(t) e^{-i\omega t} \, dt.$$

Then the power spectral density can be defined as

$$S_{xx}(\omega) = \lim_{T\to\infty} \mathbf{E}\left[|\hat{x}_T(\omega)|^2\right].$$

Here **E** denotes the expected value; explicitly, we have

$$\mathbf{E}\left[|\hat{x}_T(\omega)|^2\right] = \mathbf{E}\left[\frac{1}{T}\int_0^T x^*(t)e^{i\omega t}\,dt \int_0^T x(t')e^{-i\omega t'}\,dt'\right] = \frac{1}{T}\int_0^T\int_0^T \mathbf{E}\left[x^*(t)x(t')\right] e^{i\omega(t-t')}\,dt\,dt'.$$

Using such formal reasoning, one may already guess that for a stationary random process, the power spectral density $f(\omega)$ and the autocorrelation function of this signal $\gamma(\tau) = \langle X(t)X(t+\tau)\rangle$ should be a Fourier transform pair. Provided that $\gamma(\tau)$ is absolutely integrable, which is not always true, then

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} \gamma(\tau)\, e^{-i\omega\tau}\, d\tau = \hat{\gamma}(\omega).$$

The Wiener–Khinchin theorem makes sense of this formula for any wide-sense stationary process under weaker hypotheses: γ does not need to be absolutely integrable, it only needs to exist. But the integral can no longer be interpreted as usual. The formula also makes sense if interpreted as involving distributions (in the sense of Laurent Schwartz, not in the sense of a statistical Cumulative distribution function) instead of functions. If γ is continuous, Bochner's theorem can be used to prove that its Fourier transform exists as a positive measure, whose distribution function is F (but not necessarily as a function and not necessarily possessing a probability density).

Many authors use this equality to actually *define* the power spectral density.

The power of the signal in a given frequency band $[\omega_1, \omega_2]$ can be calculated by integrating over positive and negative frequencies,

$$\int_{\omega_1}^{\omega_2} S_{xx}(\omega) + S_{xx}(-\omega)\, d\omega = F(\omega_2) - F(-\omega_2)$$

where F is the integrated spectrum whose derivative is S_{xx}.

More generally, similar techniques may be used to estimate a time-varying spectral density.

The definition of the power spectral density generalizes in a straightforward manner to finite time-series x_n with $1 \le n \le N$, such as a signal sampled at discrete times $x_n = x(n\Delta t)$ for a total measurement period $T = N\Delta t$.

$$S_{xx}(\omega) = \frac{(\Delta t)^2}{T} \left| \sum_{n=1}^{N} x_n e^{-i\omega n} \right|^2.$$

In a real-world application, one would typically average this single-measurement PSD over several repetitions of the measurement to obtain a more accurate estimate of the theoretical PSD of the physical process underlying the individual measurements. This computed PSD is sometimes called periodogram. One can prove that this periodogram converges to the true PSD when the averaging time interval T goes to infinity (Brown & Hwang) to approach the Power Spectral Density (PSD).

If two signals both possess power spectral densities, then a cross-spectral density can be calculated by using their cross-correlation function.

Properties of the power spectral density:-

Some properties of the PSD include:

- The spectrum of a real valued process is an even function of frequency: $S_{xx}(-\omega) = S_{xx}(\omega)$.
- If the process is continuous and purely in deterministic, the auto covariance function can be reconstructed by using the Inverse Fourier transform
- it describes the distribution of the variance over frequency. In particular, $$\mathrm{Var}(X_n) = \gamma_0 = 2\int_0^\infty S_{xx}(\omega) d\omega.$$
- It is a linear function of the auto covariance function in the sense that if γ is decomposed into two functions $\gamma(\tau) = \alpha_1\gamma_1(\tau) + \alpha_2\gamma_2(\tau)$, then $f = \alpha_1 S_{xx,1} + \alpha_2 S_{xx,2}$.

The integrated spectrum or power spectral distribution $F(\omega)$ is defined as

$$F(\omega) = \int_{-\infty}^{\omega} S_{xx}(\omega')\, d\omega'.$$

	RGPV QUESTIONS	Year	Marks
Q.1	Discus about the parseval's power theorem.	JUNE 2013	7
Q.2	Prove the Parseval theorem for power signals.	DEC 2012	7

UNIT-01/LECTURE-08

CONVOLUTIONS CORRELATION & AUTO CORRELATION

Convolution:- **(DEC 2013)(7)**

Convolution is a mathematical operation and is useful for describing the input/output relationship is linier time invariant system.

convolution is a mathematical operation on two functions $f_1(t)$ and $f_2(t)$, producing a third function that is typically viewed as a modified version of one of the original functions, giving the area overlap between the two functions as a function of the amount that one of the original functions is translated. Convolution is similar to cross-correlation. It has applications that include probability, statistics, computer vision, image and signal processing, electrical engineering, and differential equations.

The convolution of $f_1(t)$ and $f_2(t)$ is written $f_1(t)\otimes f_2(t)$, using an asterisk or star. It is defined as the integral of the product of the two functions after one is reversed and shifted. As such, it is a particular kind of integral transform:

$$f_1(t)\otimes f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau$$

$$f_1(t)\otimes f_2(t) = \int_{-\infty}^{\infty} f_1(t-\tau)f_2(\tau)d\tau$$

(commutativity)

While the symbol *t* is used above, it need not represent the time domain. But in that context, the convolution formula can be described as a weighted average of the function $f_1(\tau)$ at the moment *t* where the weighting is given by $f_2(-\tau)$ simply shifted by amount *t*. As *t* changes, the weighting function emphasizes different parts of the input function.

Time Convolution Theorem:- **(DEC 2010)(10)**

This theorem stats that convolution in time domain is equivalent to multiplication of their spectra in frequency domain; i.e., if

$$f_1(t) \leftrightarrow F_1(\omega)$$

$$f_2(t) \leftrightarrow F_2(\omega)$$

$$f_1(t)\otimes f_2(t) \leftrightarrow F_1(\omega)F_2(\omega)$$

Proof

$$F[f_1(t)\otimes f_2(t)] = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f_1(\tau)f_2(t-\tau)d\tau\right]e^{-j\omega t}dt$$

$$F[f_1(t)\otimes f_2(t)] = \int_{-\infty}^{\infty} f_1(\tau)\left[\int_{-\infty}^{\infty} f_2(t-\tau)e^{-j\omega t}dt\right]d\tau$$

$$\int_{-\infty}^{\infty} f_2(t-\tau)e^{-j\omega t}dt = F_2(\omega)e^{-j\omega\tau} \text{ Using Time Shifting Property } f(t-b) \leftrightarrow F(\omega)e^{-j\omega b}$$

Hence

$$F[f_1(t) \otimes f_2(t)] = \int_{-\infty}^{\infty} f_1(\tau)F_2(\omega)e^{-j\omega\tau}\,d\tau = F_2(\omega)\int_{-\infty}^{\infty} f_1(\tau)e^{-j\omega\tau}\,d\tau$$

$$F[f_1(t) \otimes f_2(t)] = F_1(\omega)F_2(\omega)$$

$$f_1(t) \otimes f_2(t) \leftrightarrow F_1(\omega)F_2(\omega)$$

Frequency Convolution Theorem:- **(JUNE 2011)(10)**

This theorem stats that multiplication in time domain is equivalent to convolution of their spectra in frequency domain; i.e., if

$$f_1(t) \leftrightarrow F_1(\omega)$$

$$f_2(t) \leftrightarrow F_2(\omega)$$

$$2\pi f_1(t)f_2(t) \leftrightarrow F_1(\omega) \otimes F_2(\omega)$$

Proof

$$F^{-}[F_1(\omega) \otimes F_2(\omega)\,] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} F_1(\tau)F_2(\omega-\tau)d\tau\right]e^{j\omega t}d\omega$$

$$F^{-}[F_1(\omega) \otimes F_2(\omega)\,] = \int_{-\infty}^{\infty} F_1(\tau)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(\omega-\tau)e^{j\omega t}d\omega\right]d\tau$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F_2(\omega-\tau)e^{j\omega t}d\omega = f_2(t)\,e^{j\tau t} \text{ Using Frequency Shifting Property } f(t)e^{-j\omega_c t}\; f(\omega-\omega_c)$$

Hence

$$F^{-}\left[F_1(\omega) \otimes F_2(\omega)\,\right] = \int_{-\infty}^{\infty} F_1(\tau)\left[f_2(t)e^{j\tau t}\right]d\tau = f_2(t)\;2\pi\,\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\tau)e^{j\tau t}\,d\tau$$

where $\frac{1}{2\pi}\int_{-\infty}^{\infty} F_1(\tau)e^{j\tau t}\,d\tau = f_1(t)$

$$F^{-}\left[F_1(\omega) \otimes F_2(\omega)\,\right] = 2\pi f_1(t)f_2(t)$$

$$\left[F_1(\omega) \otimes F_2(\omega)\,\right] \leftrightarrow 2\pi\, f_1(t)f_2(t)$$

Convolution with impulse function:-

The convolution of a function f(t) with unit impulse function is given as

$$f(t)\otimes\delta(t) = \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)d\tau$$

Using the sampling property of impulse function, the right hand side yield the function $|f(\tau)|_{\tau=t}$ i.e., $f(t)$. Hence the convolution of a function f(t) with a unit impulse function results the function itself

The result can also be proved by using the time convolution theorem, according to which

We know that $f_1(t)\otimes f_2(t) \leftrightarrow F_1(\omega)F_2(\omega)$

$$so\ f(t)\otimes\delta(t) \leftrightarrow F(\omega)\delta(\omega)$$

$$f(t)\otimes\delta(t) = F^{-}[F(\omega)\delta(\omega)]$$

$$f(t)\otimes\delta(t) = F^{-}[F\{f(t)\}F\{\delta(t)\}]$$

$F\{f(t)\} = F(\omega)$ and $F\{\delta(t)\} = 1$

$$f(t)\otimes\delta(t) = F^{-}[F(\omega)]$$

Hence

$$f(t)\otimes\delta(t) = f(t)$$

Correlation or Cross-correlation:- **(DEC 2012)(7)**

Correlation determines the degree of similarity between two signals. If the signals are identical, then the correlation coefficient is 1; if they are totally different, the correlation coefficient is 0, and if they are identical except that the phase is shifted by exactly 180 (i.e. mirrored), then the correlation coefficient is -1.

Cross-correlation is the method which basically underlies implementations of the Fourier transformation: signals of varying frequency and phase are correlated with the input signal, and the degree of correlation in terms of frequency and phase represents the frequency and phase spectrums of the input signal.

In signal processing, cross-correlation is a measure of similarity of two waveforms as a function of a time-lag applied to one of them. This is also known as a sliding dot product or sliding inner-product. It is commonly used for searching a long signal for a shorter, known feature. It has applications in pattern recognition, single particle analysis, electron tomography, averaging, cryptanalysis, and neurophysiology.

Auto Correlation:-

- In an autocorrelation, which is the cross-correlation of a signal with itself, there will always be a peak at a lag of zero unless the signal is a trivial zero signal.
- When two independent signals are compared, the procedure is known as cross-correlation, and when the same signal is compared to phase shifted copies of it self, the procedure is known as autocorrelation.
- Autocorrelation is a method which is frequently used for the extraction of fundamental frequency, F_0 if a copy of the signal is shifted in phase, the distance between correlation peaks is taken to be the fundamental period of the signal (directly related to the fundamental frequency). The method may be combined with the simple smoothing operations of peak and centre clipping, or with other low-pass filter operations.
- Autocorrelation:- also known as serial correlation, is the cross-correlation of a signal with itself. Informally, it is the similarity between observations as a function of the

time lag between them. It is a mathematical tool for finding repeating patterns, such as the presence of a periodic signal obscured by noise, or identifying the missing fundamental frequency in a signal implied by its harmonic frequencies. It is often used in signal processing for analyzing functions or series of values, such as time domain signals.

Properties:-

In the following, we will describe properties of one-dimensional autocorrelations only, since most properties are easily transferred from the one-dimensional case to the multi-dimensional cases.

- A fundamental property of the autocorrelation is symmetry, $R(i) = R(-i)$, which is easy to prove from the definition. In the continuous case, the autocorrelation is an even function
  $R_f(-\tau) = R_f(\tau)$ when f is a real function,
  and the autocorrelation is a Hermitian function
  $R_f(-\tau) = R_f^*(\tau)$ when f is a complex function.
- The continuous autocorrelation function reaches its peak at the origin, where it takes a real value, i.e. for any delay τ, $|R_f(\tau)| \leq R_f(0)$. This is a consequence of the Rearrangement inequality. The same result holds in the discrete case.
- The autocorrelation of a periodic function is, itself, periodic with the same period.
- The autocorrelation of the sum of two completely uncorrelated functions (the cross-correlation is zero for all τ) is the sum of the autocorrelations of each function separately.
- Since autocorrelation is a specific type of cross-correlation, it maintains all the properties of cross-correlation.
- The autocorrelation of a continuous-time white noise signal will have a strong peak (represented by a Dirac delta function) at $\tau = 0$ and will be absolutely 0 for all other τ.
- The Wiener–Khinchin theorem relates the autocorrelation function to the power spectral density via the Fourier transform:

$$R(\tau) = \int_{-\infty}^{\infty} S(f) e^{j2\pi f\tau} \, \mathrm{d}f$$

$$S(f) = \int_{-\infty}^{\infty} R(\tau) e^{-j2\pi f\tau} \, \mathrm{d}\tau.$$

- For real-valued functions, the symmetric autocorrelation function has a real symmetric transform, so the Wiener–Khinchin theorem can be re-expressed in terms of real cosines only:

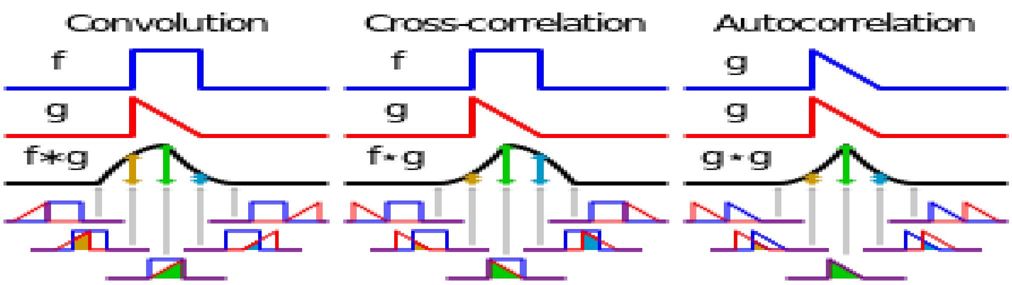
$$R(\tau) = \int_{-\infty}^{\infty} S(f) \cos(2\pi f\tau) \, \mathrm{d}f$$

$$S(f) = \int_{-\infty}^{\infty} R(\tau) \cos(2\pi f\tau) \, \mathrm{d}\tau.$$

Application:-

- One application of autocorrelation is the measurement of optical spectra and the measurement of very-short-duration light pulses produced by lasers, both using optical autocorrelators.

- Autocorrelation is used to analyze dynamic light scattering data, which notably enables determination of the particle size distributions of nanometer-sized particles or micelles suspended in a fluid. A laser shining into the mixture produces a speckle pattern that results from the motion of the particles. Autocorrelation of the signal can be analyzed in terms of the diffusion of the particles. From this, knowing the viscosity of the fluid, the sizes of the particles can be calculated.
- The small-angle X-ray scattering intensity of a nanostructured system is the Fourier transform of the spatial autocorrelation function of the electron density.
- In optics, normalized autocorrelations and cross-correlations give the degree of coherence of an electromagnetic field.
- In signal processing, autocorrelation can give information about repeating events like musical beats (for example, to determine tempo) or pulsar frequencies, though it cannot tell the position in time of the beat. It can also be used to estimate the pitch of a musical tone.
- In music recording, autocorrelation is used as a pitch detection algorithm prior to vocal processing, as a distortion effect or to eliminate undesired mistakes and inaccuracies.
- Autocorrelation in space rather than time, via the Patterson function, is used by X-ray diffractionists to help recover the "Fourier phase information" on atom positions not available through diffraction alone.
- In statistics, spatial autocorrelation between sample locations also helps one estimate mean value uncertainties when sampling a heterogeneous population.
- The SEQUEST algorithm for analyzing mass spectra makes use of autocorrelation in conjunction with cross-correlation to score the similarity of an observed spectrum to an idealized spectrum representing a peptide.
- In Astrophysics, auto-correlation is used to study and characterize the spatial distribution of galaxies in the Universe and in multi-wavelength observations of Low Mass X-ray Binaries.
- In panel data, spatial autocorrelation refers to correlation of a variable with itself through space.
- In analysis of Markov chain Monte Carlo data, autocorrelation must be taken into account for correct error determination.



Visual comparison of convolution, cross-correlation and autocorrelation. The cross-correlation is similar in nature to the convolution of two functions.

	RGPV QUESTIONS	Year	Marks
Q.1	Define convolution. What is the significance of convolution in communication system ?	DEC 2013	7

Q.2	Discuss briefly about correlation and auto correlation of signals.	DEC 2012	7
Q.3	Explain and differential between convolution, correlation and autocorrelation.	JUNE 2012	7
Q.4	State and prove the frequency convolution theorem.	JUNE 2011	10
Q.5	Define convolution. State and prove time convolution theorem in Fourier transform.	DEC 2010	10

UNIT-01/LECTURE-09

PROPERTY OF SYSTEM

A **causal system** (also known as a physical or **no anticipative system**) is a system where the output depends on past and current inputs but not future inputs i.e. the output $y(t_0)$ only depends on the input $x(t)$ for values of $t \le t_0$.

The idea that the output of a function at any time depends only on past and present values of input is defined by the property commonly referred to as causality. A system that has *some* dependence on input values from the future (in addition to possible dependence on past or current input values) is termed a non-causal or a causal system, and a system that depends *solely* on future input values is an anticausal system. Note that some authors have defined an anticausal system as one that depends solely on future *and present* input values or, more simply, as a system that does not depend on past input values.

Classically, nature or physical reality has been considered to be a causal system. Physics involving special relativity or general relativity require more careful definitions of causality, as described elaborately in causality (physics).

The causality of systems also plays an important role in digital signal processing, where filters are constructed so that they are causal, sometimes by altering a non-causal formulation to remove the lack of causality so that it is realizable. For more information, see causal filter. For a causal system, the impulse response of the system must be 0 for all $t < 0$. That is the sole necessary as well as sufficient condition for causality of a system, linear or non-linear. Note that similar rules apply to either discrete or continuous cases.

Causal and Noncausal System:- **(JUNE 2011)(5)**

A) Causal systems

Definition:- A system is said to be causal system if its output depends on present and past inputs only and not on future inputs.

Examples:- The output of casual system depends on present and past inputs, it means y(n) is a function of x(n), x(n-1), x(n-2), x(n-3)…etc. Some examples of causal systems are given below:

1) $y(n) = x(n) + x(n-2)$

2) $y(n) = x(n-1) - x(n-3)$

3) $y(n) = 7x(n-5)$

Significance of causal systems:-

Since causal system does not include future input samples; such system is practically realizable. That mean such system can be implemented practically. Generally all real time systems are causal systems; because in real time applications only present and past samples are present.

Since future samples are not present; causal system is memoryless system.

B) Anticausal or non-causal system:

Definition:

A system whose present response depends on future values of the inputs is called as a non-causal system.

Examples:

In this case, output y(n) is function of x(n), x(n-1), x(n-2)…etc. as well as it is function of x(n+1), x(n+2), x(n+3), … etc. following are some examples of non-causal systems:

1) $Y(n) = x(n) + x(n+1)$

2) $Y(n) = 7x(n+2)$

3) $Y(n) = x(n) + 9x(n+5)$

Significance:

Since non-causal system contains future samples; a non-causal system is practically not realizable. That means in practical cases it is not possible to implement a non-causal system.

- But if the signals are stored in the memory and at a later time they are used by a system then such signals are treated as advanced or future signal. Because such signals are already present, before the system has started its operation. In such cases it is possible to implement a non-causal system.
- Some practical examples of non-causal systems are as follows:

1) Population growth

2) Weather forecasting

3) Planning commission etc

Linear and Non Linear systems:

A system is said to be linear if it follows both the Homogeneity and superposition principles.

Homogeneity: If the input is multiplied by a constant, the output shall also be multiplied by the same.

Superposition: If the input is superposed by two signals, the out put shall also be superposed.

So, a general description of a linear system is

iff X1,2(n) -> Y1,2(n) => aX1(n)+bX2(n) => aY1(n)+bY2(n)

Anything, which is not a linear system, which means that it doesn't follow either of the above properties or all of them, the system is called non linear.

We can check for linearity by making X(n) equal to zero and see whether Y(n) becomes the same. If not, we can conclude it to be linear. But if it is zero, we need to further test the difference equation for superposition and then if the difference equation satisfies it, then the system is acknowledged as linear. We can sum that, any system with a non zero initial condition is a non linear system. A charged capacitor and an inductor with initial flux are all non linear.

Time Variant or Time Invariant Systems

Definition:

A system is said to be Time Invariant if its input output characteristics do not change with time. Otherwise it is said to be Time Variant system.

Explanation:

As already mentioned time invariant systems are those systems whose input output characteristics do not change with time shifting. Let us consider x(n) be the input to the system which produces output y(n) as shown in figure below.

Now delay input by k samples, it means our new input will become x(n-k). Now apply this delayed input x(n-k) to the same system as shown in figure below.

Now if the output of this system also delayed by k samples (i.e. if output is equal to y(n-k)) then this system is said to be Time invariant (or shift invariant) system.

If we observe carefully, x(n) is the initial input to the system which gives output y(n), if we delayed input by k samples output is also delayed by same (k) samples. Thus we can say that input output characteristics of the system do not change with time. Hence it is Time invariant system.

Theorem:

A system is Time Invariant if and only if

$x(n) \xrightarrow{T} y(n)$ implies that $x(n-k) \rightarrow y(n-k)$

Similarly a continuous time system is Time Invariant if and only if

$x(t) \xrightarrow{T} y(t)$ implies that $x(t-k) \rightarrow y(t-k)$

Now let us discuss about How to determine that the given system is Time invariant or not?

To determine whether the given system is Time Invariant or Time Variant, we have to follow the following steps:

Step 1: Delay the input x(n) by k samples i.e. x(n-k). Denote the corresponding output by y(n,k).

That means $x(n-k) \rightarrow y(n,k)$

Step 2: In the given equation of system y(n) replace 'n' by 'n-k' throughout. Thus the output is y(n-k).

Step 3: If y(n,k) = y(n-k) then the system is time invariant (TIV) and if y(n,k) ≠ y(n-k) then system is time variant (TV).

Same steps are applicable for the continuous time systems.

Solved Problems:

1) Determine whether the following system is time invariant or not.

y(n) = x(n) – x(n-2)

Solution:

Step 1: Delay the input **y** 'k' samples and denote the output by y(n,k)

Therefore y(n,k) = x(n-k) – x(n-2-k)

Step 2: Replace 'n' by 'n-k' throughout the given equation.

Therefore y(n-k) = x(n-k) – x(n-k-2)

Step 3: Compare above two equations. Here y(n,k) = y(n-k). Thus the system is Time Invariant.

2) Determine whether the following systems are time invariant or not?

y(n) = x(n) + n x(n-2)

Solution:

Step 1: Delay the input by 'k' samples and denote the output by y(n,k)

Therefore y(n,k) = x(n-k) + n x(n-2)

Step 2: Replace 'n' by 'n-k' throughout the given equation.

Therefore y(n-k) = x(n-k) + (n-k) x(n-k-2)

Step 3: Compare above two equations. Here y(n,k) ≠ y(n-k). Thus the system is Time Variant.

	RGPV QUESTIONS	Year	Marks
Q.1	Explain causal and non-causal system in short.	JUNE 2011	5