Unit:01

Signal Analysis: Time domain and frequency domain representation of signal, Fourier Transform and its properties, Transform of Gate, Periodic gate, Impulse periodic impulse sine and cosine wave, Concept of energy density and power, Power density of periodic.

1.1 Time domain and frequency domain representation of signal

An electrical signal either, a voltage signal or a current signal can be represented in two forms: These two types of representations are as under:

i) Time Domain representation-: In time domain representation a signal is a time varying quantity as shown in Fig.1.1

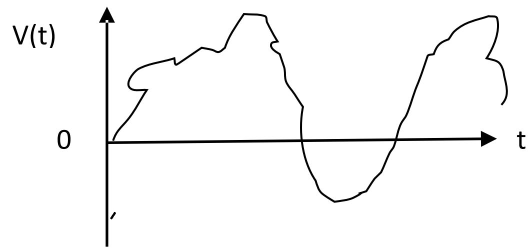

Fig 1.1 An arbitrary time domain signal

ii) Frequency Domain Representation: In frequency domain, a signal is represented by its frequency spectrum as shown in Fig 1.2

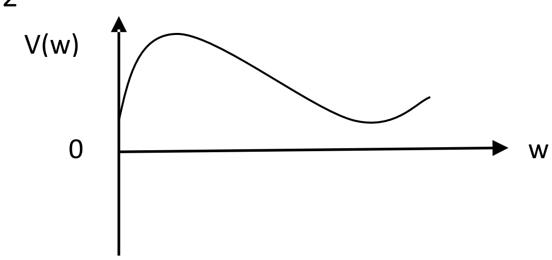

Fig 1.2 Frequency domain representation of time domain signal

1.2 Fourier Transform and its properties

Fourier Transform pair

Fourier transform may be expressed as

$$X(w)=F[x(t)]=\int_{-\infty}^{\infty} x(t)\, e^{-jwt} dt$$

In the above equation X(w) is called the Fourier transform of x(t). In other words X(w) is the frequency domain representation of time domain function x(t). This means that we are converting a time domain signal into its frequency domain representation with the help of fourier transform. Conversely if we want to convert frequency domain signal into corresponding time domain signal, we will have to take inverse fourier transform of frequency domain signal. Mathematically, Inverse fourier transform.

$$F^{-1}[X(w)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}dw$$

Example

Q.1 Find the fourier transform of a single-sided exponential function $e^{-at}u(t)$.

Solution: $e^{-at}u(t)$ is single sided function because her the main function e^{-at} is multiplied by unit step function u(t), then resulting signal will exist only for t>0.

u(t)= {1 for t>1

={0 for elsewhere

Now, given that x(t)= $e^{-at}u(t)$

X(w)=F[x(t)]=$\int_{-\infty}^{\infty} x(t)\, e^{-jwt}$dt

Or X(w)=$\int_{-\infty}^{\infty} e^{-at}\, u(t)\, e^{-jwt}$dt

=$\int_{0}^{\infty} e^{-at}\ e^{-jwt}$dt

=$\int_{0}^{\infty} e^{-t(a+jw)}$dt

$$\frac{-1}{(a+jw)}[e^{-\infty} - e^{0}] = \frac{-1}{(a+jw)}[0\text{-}1] = \frac{1}{(a+jw)}$$

To obtain the above expression in the proper form we write

X(w)= $\frac{-1}{(a+jw)} * \frac{(a-jw)}{(a-jw)}$

X(w)= $\frac{(a-jw)}{(a^2+w^2)} = \frac{a}{(a^2+w^2)} - \frac{jw}{(a^2+w^2)}$

Obtaining the above expression in polar form

X(w)=$\frac{1}{\sqrt{a^2+w^2}}\, e^{-jtan^{-1}(\frac{w}{a})}$

As we know that

X(w)=$|X(w)|e^{j\varphi(jw)}$

On comparision amplitude spectrum

$|X(w)|$=$\frac{1}{\sqrt{a^2+w^2}}$

$$\varphi(w) = -tan^{-1}(\frac{w}{a})$$

Properties of Continuous Time Fourier Transform (CTFS)

1. **Time Scaling Function**

   Time scaling property states that the time compression of a signal results in its spectrum expansion and time expansion of the signal results in its spectral compression. Mathematically,

   If x(t) $\longleftrightarrow$ X(w)

   Then, for any real constant a,

$x(at) \longleftrightarrow \frac{1}{|a|}X(\frac{w}{a})$

proof: The general expression for fourier transform is

$X(w)=F[x(t)]=\int_{-\infty}^{\infty} x(t)\, e^{-jwt}dt$

Now $F[x(at)]=\int_{-\infty}^{\infty} x(at)\, e^{-jwt}dt$

Putting

At=y

We have $dt=\frac{dy}{a}$

Case (i): When a is positive real constant

$$F[x(at)]= \int_{-\infty}^{\infty} x(y)\, e^{-j\left(\frac{w}{a}\right)y}\, \frac{dy}{a} = \frac{1}{a}\int_{-\infty}^{\infty} x(y)\, e^{-j\left(\frac{w}{a}\right)y} = \frac{1}{a}X(\frac{w}{a})$$

Case (ii): When a is negative real constant

$$F[x(at)]= \frac{-1}{a}X(\frac{w}{a})$$

Combining two cases, we have

$$F[x(at)]=\frac{1}{|a|}X(\frac{w}{a}) \text{ Or } x(at) \longleftrightarrow \frac{1}{|a|}X(\frac{w}{a})$$

The function x(at) represents the function x(t) compressed in time domain by a factor a. Similarly, a function $X(\frac{w}{a})$ represents the function X(w) expanded in frequency domain by the same factor a.

2. **Linearity Property**

Linearity property states that fourier transform is linear. This means that

If $x_1(t) \longleftrightarrow X_1(w)$

And $x_2(t) \longleftrightarrow X_2(w)$

Then $a_1 x_1(t) + a_2 x_2(t) \longleftrightarrow a_1X_1(w) + a_2X_2(w)$

3. **Duality or Symmetry Property**

If $x(t) \longleftrightarrow X(w)$

Then $X(t) \longleftrightarrow 2\pi x(-w)$

Proof

The general expression for fourier transform is

$$F^{-1}[X(w)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}dw$$

Therefore,

$$x(-t)= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{-jwt}dw$$

$2\pi x(-t)= \int_{-\infty}^{\infty} X(w)e^{-jwt}dw$

Since w is a dummy variable, interchanging the variable t and w we have

$2\pi x(-w)= \int_{-\infty}^{\infty} X(t)e^{-jwt}dw$=F[X(t)]

Or F[X(t)]= 2πx(-w)

Or X(t) $\longleftrightarrow$ 2πx(-w)

For an even function x(-w)=x(w)

Therefore , X(t) $\longleftrightarrow$ 2πx(w)

Example (1)

The fourier transform F[$e^{-t}u(t)$] is equal to $\frac{1}{1+j2\pi f}$. Therefore F[$\frac{1}{1+j2\pi f}$] is equal to

Solution:

Using Duality property of Fourier Transform, we have

If x(t) $\longleftrightarrow$ X(f)

Then X(t) $\longleftrightarrow$ x(-f)

Therefore,

$e^{-t}u(t) \longleftrightarrow \frac{1}{1+j2\pi f}$

Then $\frac{1}{1+j2\pi t} \longleftrightarrow e^{-f}u(f)$

4. **Time Shifting property**

Time Shifting property states that a shift in the time domain by an amount b is equivalent to multiplication by e^{-jwb} in the frequency domain. This means that magnitude spectrum $|X(w)|$

Remains inchanged but phase spectrum θ(w) is changed by -wb.

If x(t) $\longleftrightarrow$ X(w)

Then X(t-b) $\longleftrightarrow$ X(w) e^{-jwb}

Proof: X(w)=F[x(t)]=$\int_{-\infty}^{\infty} x(t)\, e^{-jwt}$dt

And F[x(t-b)]= $\int_{-\infty}^{\infty} x(t-b)\, e^{-jwt}$dt

Putting t-b=y, so that dt=dy

F[x(t-b)]= $\int_{-\infty}^{\infty} x(y)\, e^{-jw(b+y)}$dy =$\int_{-\infty}^{\infty} x(y)\, e^{-jwb}e^{-jwy}$ dy

Or F[x(t-b)]= $e^{-jwb} \int_{-\infty}^{\infty} x(y)\, e^{-jwy}$ dy

Since y is a dummy variable, we have

F[x(t-b)]= e^{-jwb}X(w)=X(w) e^{-jwb}

Or x(t-b) $\longleftrightarrow$ X(w) e^{-jwb}

5. **Frequency Shifting Property**

Frequency shifting property states that the multiplication of function x(t) by e^{jw_0t} is equivalent to shifting its fourier transform X(w) in the positive direction by an amount w_0 . This means that the spectrum X(w) is translated by an amount c. hence this property is often called frequency translated theorem. Mathematically .

If x(t) $\longleftrightarrow$ X(w)

Then e^{jw_0t} x(t) $\longleftrightarrow$ X(w-w_0)

Proof: General expression for fourier transform is

X(w)=F[x(t)]=$\int_{-\infty}^{\infty} x(t)\, e^{-jwt}$dt

Now, F[e^{jw_0t}x(t)] = $\int_{-\infty}^{\infty} x(t)\, e^{jw_0t}e^{-jwt}$dt

Or F[e^{jw_0t}x(t)] = $\int_{-\infty}^{\infty} x(t)\, e^{-j(w-w_0)t}$dt

Or F[e^{jw_0t}x(t)] = $\mathrm{X}(\mathrm{w}-w_0)$

Or e^{jw_0t} x(t) $\longleftrightarrow$ X(w-w_0)

6. **Time Differentiation Property**

The time differentiation property states that the differentiation of a function x(t) in the time domain is equivalent to multiplication of its fourier transform by a factor jw. Mathematically

If x(t) $\longleftrightarrow$ X(w)

Then $\frac{dx(t)}{dt}$ x(t) $\longleftrightarrow$ jw X(w)

Proof: The general expression for fourier transform is

$$F^{-1}[X(w)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}dw$$

Taking differentiation, we have

$\frac{dx(t)}{dt}$ =$\frac{1}{2\pi}\frac{d}{dt}[\int_{-\infty}^{\infty} X(w)e^{jwt}dw]$

Interchanging the order of differentiation and integration, we have

$\frac{dx(t)}{dt}$ =$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d}{dt}[X(w)e^{jwt}]dw$

=$\frac{1}{2\pi}\int_{-\infty}^{\infty} jw\, X(w)e^{jwt}dw$

Or $\frac{dx(t)}{dt} = F^{-1}[jwX(w)]$

Or $F[\frac{dx(t)}{dt}] = jwX(w)$

Or $\frac{dx(t)}{dt} \longleftrightarrow jwX(w)$ Hence proved

1.3 Transform of Gate

A gate function is rectangular pulse. Figure 1.3 shows gate function. The function or rectangular pulse shown in figure 1.3 is written as rect $(\frac{t}{\tau})$.

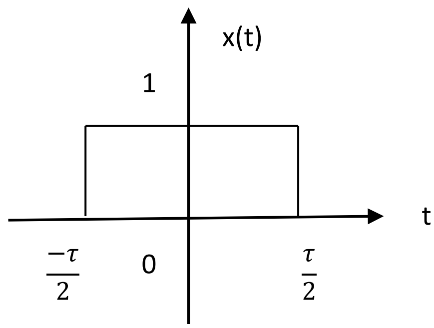

Fig.1.3 A Gate Function

From the above figure it is clear that rect $(\frac{t}{\tau})$ represents a gate pulse of height or amplitude unity and width τ.

x(t)= rect $(\frac{t}{\tau})$ $=\left\{1 \quad for \ \frac{-\tau}{2} < t < \frac{\tau}{2}\right\}$

$\{0 \quad otherwise\}$

Sampling Function Or Interpolation Function Or Sinc function

The functions $\frac{sinx}{x}$ is the "sine over argument" and denoted by sinc(x). This function plays an important role in signal processing. It is also known as the filtering or interpolating function. Mathematically,

Sinc(x)=$\frac{sinx}{x}$

Or

Sa(x)= $\frac{sinx}{x}$

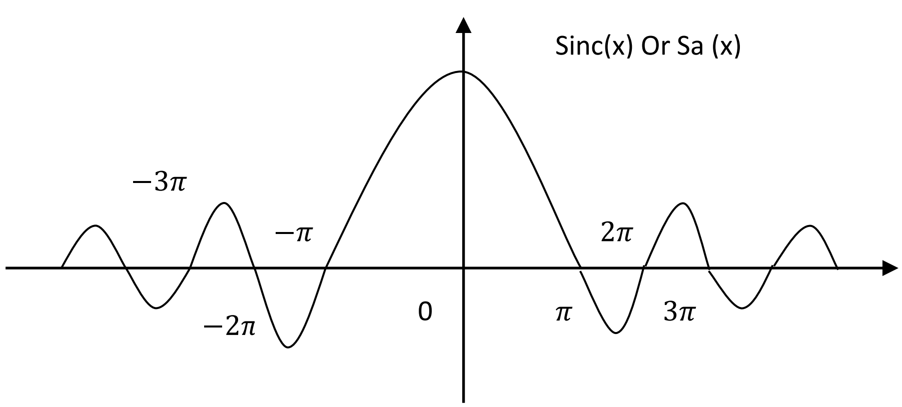

Fig.1.4 Sample function

From the figure, following points may be observed about the sampling function :

(i) Sa(x) or sinc(x) is an even function of x.

(ii) Sinc(x) =0 when sinx=0 except at x=0, where it is indeterminate. This means that sinc(x)=0 for x=$\pm$nπ , here n=$\pm 1, \pm 2$ ....

(iii) Sinc(x) is the product of oscillating signal sinx of period 2π and a decreasing function $\frac{1}{x}$. Therefore, sinc(x) exhibits sinusoidal oscillations of period 2π with amplitude decreasing continuously as 1/x.

Example 2: Find the fourier transform of the gate function shown in figure 1.5.

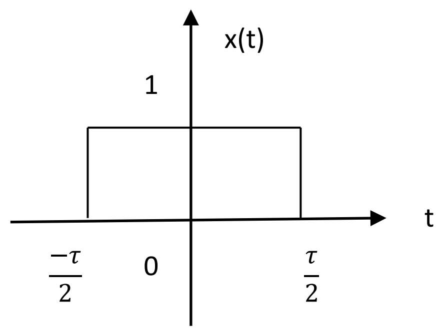

Fig.1.5

Sol. x(t)= rect $(\frac{t}{\tau})$ =$\{1 \quad for \ \frac{-\tau}{2} < t < \frac{\tau}{2}\}$

$\{0 \quad otherwise\}$

X(w)=F[x(t)]=$\int_{-\infty}^{\infty} x(t)\, e^{-jwt}$dt

X(w)=F[x(t)]=$\int_{-\infty}^{\infty}$ rect $(\frac{t}{\tau})$ e^{-jwt}dt

=$\int_{\frac{-\tau}{2}}^{\frac{\tau}{2}} 1.e^{-jwt}$dt= $\left[\frac{-e^{-jwt}}{jw}\right]_{\frac{-\tau}{2}}^{\frac{\tau}{2}}$

=$\frac{-1}{jw}[e^{-j\frac{w\tau}{2}}$-$e^{j\frac{w\tau}{2}}]$

=$\frac{1}{jw}[e^{j\frac{w\tau}{2}}$-$e^{-j\frac{w\tau}{2}}]$ --------(1)

We know that $e^{j\theta}$=cosθ +jsinθ

And $e^{-j\theta}$=cosθ –jsinθ

Hence $2\cos\theta = e^{j\theta} + e^{-j\theta}$

$2j\sin\theta = e^{j\theta} - e^{-j\theta}$

Putting $\theta = \frac{w\tau}{2}$, we get

$2j\sin\frac{w\tau}{2} = e^{j\frac{w\tau}{2}} - e^{-j\frac{w\tau}{2}}$ --------(2)

From (1) and (2)

$X(w) = \frac{1}{jw}[2j\sin\frac{w\tau}{2}]$

By multiplying and dividing the equation by τ

$= \frac{2\tau}{jw\tau}[j\sin\frac{w\tau}{2}]$

$= \frac{\tau}{\frac{w\tau}{2}}[\sin\frac{w\tau}{2}]$

$= \tau\ [\frac{\sin\frac{w\tau}{2}}{\frac{w\tau}{2}}]$

$= \tau \text{sinc}(\frac{w\tau}{2})$

Now, since sinc(x)=0, when $x=\pm n\pi$

Therefore, $\text{sinc}(\frac{w\tau}{2}) = 0$, when $\frac{w\tau}{2} = \pm n\pi$

Or $w = \frac{=\pm 2n\pi}{\tau}$

Figure 1.5 shows the plot of X(w) $\frac{2\pi}{\tau}$

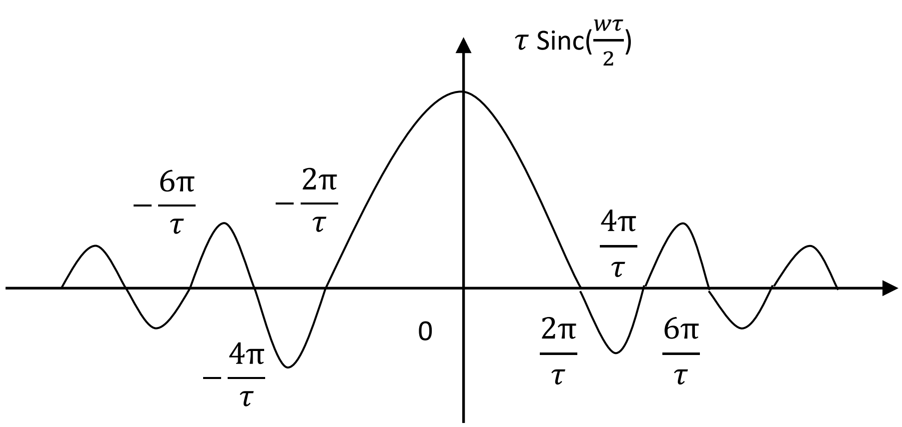

Fig. 1.5 Sample Function

1.4 Impulse Functions

Unit Impulse functions:

A unit impulse function was invented by P.A.M. Diarc and so it is also called as Delta function. It is denoted by $\delta(t)$.

Mathematically,

$\delta(t)$=0 , t$\neq$ 0

And, $\int_{-\infty}^{\infty}\delta(t)$dt=1

Figure 1.6 shows the graphical representation of an unit impulse function. The following points may be observed about an unit-impulse function:

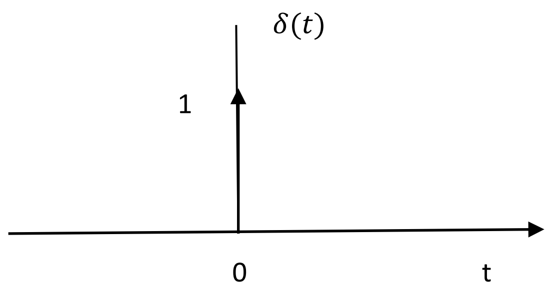

Fig.1.6 The Unit Impulse function

i) The width of pulse is zero. This means that pulse exist only at t=0.
ii) The height of the pulse goes to infinity
iii) The area under the pulse-curve is always is always unity.

Shifting Property of the Impulse function:

If we take the product of unit impulse function $\delta(t)$ and any given function x(t) which is continuous at t=0,then this product will provide the function x(t) existing only at t=0 since $\delta(t)$ exist only at t=0. Mathematically,

$$\int_{-\infty}^{\infty}x(t)\delta(t)\text{dt}=\text{x}(0)\int_{-\infty}^{\infty}\delta(t)\text{dt}=\text{x}(0).1=\text{x}(0)$$

The equation is also known as shifting or sampling property of the impulse function because the impulse shifts the value of x(t0 at t=0. This means that the value of x(t) has been sampled at t=0. to The shifting or sampling may be also done at any, instant t=t_0, if we define the impulse function at the instant. Mathematically,

$$\int_{-\infty}^{\infty}x(t)\delta(t-t_0)\text{dt}=\text{x}(t_0)$$

The above equation states that the product of a continuous function x(t) with an impulse function $\delta(t-t_0)$ provides the sampled value of x(t) at t=t_0.

Q.1 Find the fourier transform of an impulse function x(t) = $\delta(t)$ Also draw the spectrum

Sol. Expression of the fourier transform is given by

$\text{X(w)=F[x(t)]}=\int_{-\infty}^{\infty}x(t)\,e^{-jwt}\text{dt}=\int_{-\infty}^{\infty}\delta(t)\,e^{-jwt}\text{dt}$

Using shifting property of impulse function

$\text{X(w)}=[\,e^{-jwt}]_{\text{at t=0}}$

$X(w)=1$

$\delta(t) \longleftrightarrow 1$

Hence the fourier transform of an impulse function is unity.

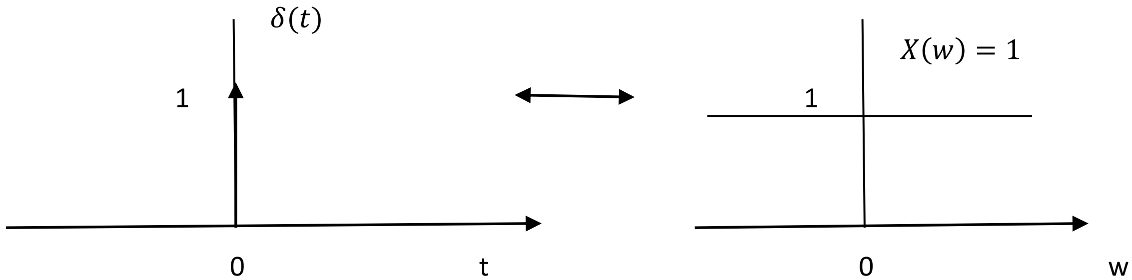

Fig 1.7

Figure 1.7 shows an unit impulse function and its fourier transform or spectrum. From the figure1.7 it is clear that an unit impulse contains the entire frequency components having identical magnitude. This means that the bandwidth of the unit impulse function is infinite. Also, since spectrum is real, only magnitude spectrum is required. The phase spectrum $\theta(w)$=0, which means that all the frequency components are in the same phase.

Q.(2) Find the inverse fourier transform of δ(w)

Solution. Inverse fourier transform is expressed as

$$F^{-1}[X(w)] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}dw$$

$$F^{-1}[\delta(\mathrm{w})] = x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\mathrm{w})e^{jwt}dw$$

$F^{-1}[\delta(\mathrm{w})]=\frac{1}{2\pi}[\ e^{jwt}]_{\text{at w=0}}$

$F^{-1}[\delta(\mathrm{w})]=\frac{1}{2\pi}[\ e^{0}]= \frac{1}{2\pi}.1\ = \frac{1}{2\pi}$

$F[\frac{1}{2\pi}]= \delta(\mathrm{w})$

$\frac{1}{2\pi} \longleftrightarrow \delta(\mathrm{w})$

$1 \longleftrightarrow 2\pi\delta(\mathrm{w})$

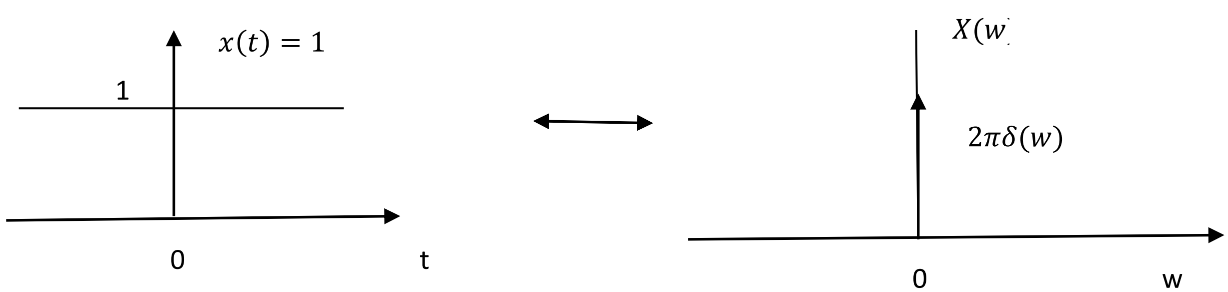

Fig.1.8

This shows that the spectrum of a constant signal x(t)=1 an impulse function $2\pi\delta(w)$. This can also be interpreted as that x(t) =1 is a d.c. signal which has single frequency. W=0(dc).

Q.(3) Find the inverse fourier transform of δ(w-w_0)

Solution. Inverse fourier transform is expressed as

$$F^{-1}[X(w)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(w)e^{jwt}dw$$

Or

$$F^{-1}[\delta(\mathrm{w}-w_0)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\mathrm{w}-w_0)e^{jwt}dw$$

Using shifting or sampling property of impulse function, we get

$F^{-1}[\delta(\mathrm{w}-w_0)]$=$\frac{1}{2\pi}$[e^{jwt}]$_{\text{at}}$ $\mathrm{w}=w_0$

$F^{-1}[\delta(\mathrm{w}-w_0)]$=$\frac{1}{2\pi}$[e^{jw_0t}]

F[$\frac{1}{2\pi}$ e^{jw_0t}]= $\delta(\mathrm{w}-w_0)$

$\frac{1}{2\pi}$ e^{jw_0t} $\longleftrightarrow$ $\delta(\mathrm{w}-w_0)$

Or

e^{jw_0t} $\longleftrightarrow$ $2\pi\ \delta(\mathrm{w}-w_0)$

The above expression shows that the spectrum of an overlasting exponential $e^{jw_0t}$is a single impulse at w=0.

Similarly,

e^{-jw_0t} $\longleftrightarrow$ $2\pi\ \delta(\mathrm{w}+w_0)$

1.5 Fourier Transform of Cosine wave

Q.4 Find the fourier transform of overlasting sinusoid cosw_0t.

Solution: We know that Euler's identity is given by

$e^{j\theta}$=cosθ +jsinθ

And , $e^{-j\theta}$=cosθ –jsinθ

Hence 2cosθ=$e^{j\theta}$+$e^{-j\theta}$ Or

cosθ=$\frac{e^{j\theta}+e^{-j\theta}}{2}$

And

$$2j\sin\theta = e^{j\theta} - e^{-j\theta} \text{ Or}$$

$\sin\theta = \frac{e^{j\theta} + e^{-j\theta}}{2j}$

Hence, $\cos w_0 t = \frac{e^{jw_0 t} + e^{-jw_0 t}}{2}$

We know that

$$e^{jw_0 t} \longleftrightarrow 2\pi\,\delta(\mathrm{w} - w_0)$$

$$And \quad e^{-jw_0 t} \longleftrightarrow 2\pi\,\delta(\mathrm{w} + w_0)$$

So that $\cos w_0 t \longleftrightarrow \frac{1}{2}[2\pi\,\delta(\mathrm{w} - w_0) + 2\pi\,\delta(\mathrm{w} + w_0)]$

Or

$\cos w_0 t \longleftrightarrow [\pi\,\delta(\mathrm{w} - w_0) + \pi\,\delta(\mathrm{w} + w_0)]$

1.6 Fourier Transform of Periodic Function

Fourier transform of periodic function could also be found out. This means that Fourier transform may be used as a universal mathematical tool to analyze both periodic and non-periodic waveform over the entire interval. Let us find the fourier transform of periodic function x(t). x(t) may be expressed in terms of complex fourier series as

$x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t}$

Taking fourier transform of both the side

$F[x(t)] = F[\sum_{n=-\infty}^{\infty} C_n e^{jnw_0 t}] \qquad = \sum_{n=-\infty}^{\infty} C_n . F[1. e^{jnw_0 t}]$

Using frequency shifting shifting theorem, we can write

$F[1. e^{jnw_0 t}] = 2\pi\,\delta(\mathrm{w} - nw_0)$

Hence, $F[x(t)] = \sum_{n=-\infty}^{\infty} C_n 2\pi\,\delta(\mathrm{w} - nw_0) = 2\pi \sum_{n=-\infty}^{\infty} C_n\,\delta(\mathrm{w} - nw_0)$

Hence, the fourier transform of a periodic function consist of a train of equally spaced impulses. These impulses are located at the harmonic frequencies of the signal and the strength or area of each impulse is given by $2\pi C_n$.

1.7 Concept of Energy Density

An energy signal is one which has finite energy and zero average power. Hence, x(t) is an energy signal if

$0 < E < \infty$ and P=0

Where, E is the energy and P is the power of the signal x(t). All the practical non-periodic signals which are defined over finite-time (also called time limited signals) are energy signals.

For continuous time signals energy of the signal is expressed as

$E = \int_{-\infty}^{\infty} x^2(t)dt$

Energy Spectral Density:

Let us consider an low pass signal which is applied to an ideal low pass filter as shown in figure 1.9.Figure 1.10 shows the graph of transfer function H(w) of an ideal low pass filter. The response of output of a system is expressed as

Y(w)=X(w)H(w)
Here X(w)= Fourier transform of x(t)
Y(w)= Fourier transform of y(t)

Also the energy E_0 of the output signal y(t) may be expressed as (using Parseval's theorem)

$$E_0=\frac{1}{2\pi}\int_{-\infty}^{\infty}|Y(w)|^2 dw$$

Or

$$E_0=\frac{1}{2\pi}\int_{-wm}^{wm}|H(w)X(w)|^2 dw$$

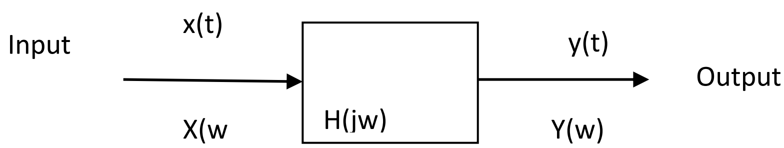

Fig.1.9 Low pass filter (LPF)

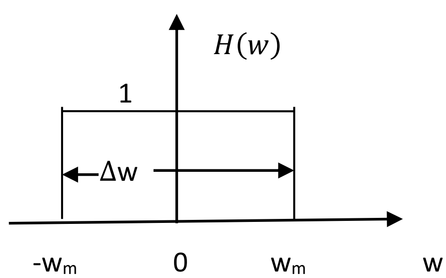

Fig.1.10 Transfer Function of an ideal low pass filter

From figure 1.10 it can be observed that H(w)=0 for all the frequencies except for the narrow band $-w_m$ to w_m .Therefore, the energy of the signal over this narrow band $\Delta w=2w_m$. will be

$$E_0=\frac{1}{2\pi}|X(w)|^2\int_{-wm}^{wm}1.dw=\frac{1}{2\pi}|X(w)|^2 2w_m$$

Putting $2w_m=\Delta w$ we get

$$E_0=\frac{1}{2\pi}.|X(w)|^2\Delta w=|X(w)|^2\Delta f \quad \text{-------(1)}$$

From the above equation it is clear that E_0 represents the contribution of energy due to the bandwidth (Δw) of the signal including negative frequencies. Therefore energy contribution per unit bandwidth will be.

$\frac{E_0}{\Delta f}=|X(w)|^2$ ------(2)

Hence, $|X(w)|^2$ represents energy per unit bandwidth, and is known as Energy spectral density Or Energy density spectrum.

It is generally denoted by Ψ(w).

Hence , $\Psi(w)= |X(w)|^2$ ----(3)

Now, we can find the relationship between energy densities of input and output (response) as under:

Since, we know that Y(w)=H(w).X(w)

Therefore, $|Y(w)|^2=|H(w)X(w)|^2=|H(w)|^2|X(w)|^2$ ------(4)

Now, let $\Psi_y(w)$ be energy spectral density of output y(t) and $\Psi_x(w)$ be energy spectral density of input x(t), then

$\Psi_y(w)= |Y(w)|^2$ ------------(5)

$\Psi_x(w)= |X(w)|^2$ ---------(6)

From equation (4) and(5)

$\Psi_y(w)= |H(w)|^2|X(w)|^2$ --------(7)

From equation (6) and(7)

$\Psi_y(w)= |H(w)|^2\Psi_x(w)$--------(7)

This is the relationship between input and output spectral densities.

Properties of Energy Spectral density function

Property1: Total area under energy spectral density function is equal to the total energy of that signal

Mathematically,

$$E=\int_{-\infty}^{\infty} \Psi(f)df$$

Property2: If x(t) is input to a linear time invariant (LTI) system with transfer function H(w) , then input and output energy spectral density function are related as

$$\Psi_o(w)= |H(w)|^2\Psi_i(w)$$

Where,

$\Psi_o(w)$= output energy spectral density function

$\Psi_i(w)$= input energy spectral density function

$|H(w)|^2$=energy gain at frequency w.

Property3: The autocorrelation function R(τ) and energy spectral density Ψ(w) form a fourier transform pair.

Mathematically,

$$R(\tau) \longleftrightarrow \Psi(w).$$

1.8 Concept of Power Density

A power signal is one which has finite average power and infinite energy. Hence x(t) is a power signal if $0<P<\infty$

And $E=\infty$

Where P is the average power and E is the energy of the signal.

Almost all practical signals are power signals since their average power is finite and non-zero. For continuous time signal, the average power P is defined as

$$P = \lim_{T\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x^2(t)dt$$

The Power Spectral Density

The expression for power spectral density may be derived by assuming the power signal a limiting case of an energy signal, such that it is zero outside the interval $\pm\frac{\tau}{2}$. Let this terminated signal be denoted as x_τ(t) may be expressed as

$$x_\tau(t)=\begin{cases} x(t) & |t| < \frac{\tau}{2} \\ 0 & elsewhere \end{cases}$$

Now since terminated signal x_τ(t) is of finite duration τ, Therefore it is an energy signal. Let the energy of this signal be denoted by E_τ and may be expressed as

$$E_\tau=\int_{-\infty}^{\infty}|x_\tau(t)|^2dt=\int_{-\infty}^{\infty}|x_\tau(w)|^2df$$

Here, x_τ(w) is th fourier transform of x_τ(t). It may be observed that x(t) over the interval $(-\frac{\tau}{2},\frac{\tau}{2})$ will be same as x_τ(t) over the interval (-∞, ∞).

Therefore, we have

$$\int_{-\infty}^{\infty}|x_\tau(t)|^2dt=\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}|x(t)|^2dt$$

Therefore,

$$\frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}|x(t)|^2dt = \frac{1}{\tau}\int_{-\infty}^{\infty}|x_\tau(w)|^2df$$

Taking τ→∞ of both sides of the equation

$$\lim_{\tau\to\infty} \frac{1}{\tau}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}}|x(t)|^2dt = \lim_{\tau\to\infty} \frac{1}{\tau}\int_{-\infty}^{\infty}|x_\tau(w)|^2df$$

The left Hand side of the above equation represents the average power P of the function x(t). Therefore

$$P= \int_{-\infty}^{\infty}\lim_{\tau\to 0}\frac{|x_\tau(w)|^2}{\tau}df \text{ ------(1)}$$

In the limit $\tau \to 0$, the ratio $\frac{|x_\tau(w)|^2}{\tau}$ may be approach a finite value.

Let this finite value be S(w).

So that $S(w)= \lim_{\tau\to 0}\frac{|x_\tau(w)|^2}{\tau}$ --------(2)

From equation (1) and (2)

$$P=\int_{-\infty}^{\infty}\lim_{\tau\to 0} S(w)df = \frac{1}{2\pi}\int_{-\infty}^{\infty}\lim_{\tau\to 0} S(w)dw$$

According to the above equation, the total power of the signal is obtained by multiplying S(w) with bandwidth Δw(dw) and integrating over the bandwidth. Hence, S(w) may be treated as average power per unit bandwidth and so is called as power spectral density Or power density spectrum.

Since $|x_\tau(w)|^2 = |x_\tau(-w)|^2$

Therefore, the power contribution by positive and negative frequencies is identical. Thus, average power is expressed as

$$P=\int_{-\infty}^{\infty} S(w)\,df = \int_{0}^{\infty} S(w)\,df = \frac{1}{\pi}\int_{0}^{\infty} S(w)\,dw$$

Properties of power spectral density function

Property1: The area under power density function is equal to the average power of that signal.

$$P=\int_{-\infty}^{\infty} S(w)\,df$$

Property2: If x(t) is input to a linear time invariant (LTI) system with transfer function H(jw) , then input and output power spectral density function are related as

$S_o(w) = |H(w)|^2 S_i(w)$

Where,

$S_o(w)$= output power spectral density

$S_i(w)$= input power spectral density

$|H(w)|^2$= power gain at frequency w.

Property3: The autocorrelation function R(τ) and power spectral density S(w) form a fourier transform pair. Mathematically,

$$R(\tau) \longleftrightarrow S(w).$$