UNIT-II
Z-TRANSFORM

The Z-Transform

The direct z-transform, properties of the z-transform, rational z-transforms, inversion of the z transform, analysis of linear time-invariant systems in the z- domain, block diagrams and signal flow graph representation of digital network, matrix representation.

2.1 DEFINITION OF Z TRANSFORMS:

The z-transform of a discrete signal x (n) is defined as the power series,

$$Z[x(n)] = X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Where z is a complex variable. This is generally referred to as two sided z-transform.

If x (n) is a causal sequence, x (n) = 0, for n < 0, then its z-transform is

$$X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

This expression is called a one sided z-transform.

This causal sequence produces negative powers of z in X (z). Generally we assume that x (n) is a causal sequence, unless it is otherwise stated.

If x (n) is non-causal sequence, x(n)=0 for 0 ≤ n, then its z-transform is

$$X(z) = \sum_{n=-\infty}^{-1} x(n)\, z^{-n}$$

This expression is also called a one sided z-transform. This non causal sequence produces positive powers of z in X (z).

2.2 REGION OF CONVERGENCE:

The values of z for which X (z) is finite and lie within the region called as "region of convergence (ROC). Therefore, the condition for X (z) to be finite is |z| > 1. In other words, the ROC for X (z) is the area outside the unit circle in the z plane.

The ROC of a rational z transform is bounded by the location of its pole. For Example, the z transform of the unit step response u(n) is $X(z) = \frac{z}{z-1}$ which has a zero at z = 0 and a pole at z = 1 and the ROC is |z| > 1 and extending all the way to ∞, as shown in figure 1

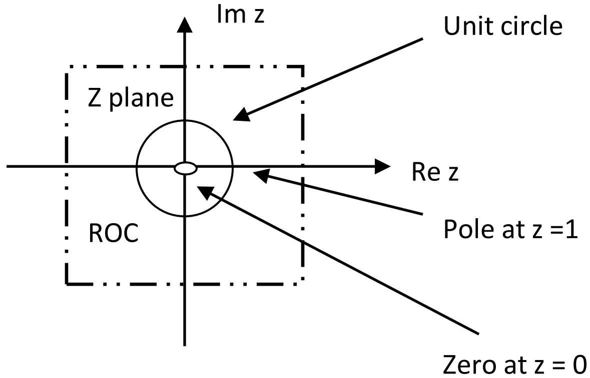


Figure 1: Pole Zero plot and ROC of Unit Step Response

PROPERTIES OF ROC OF Z-TRANSFORMS:

- X (z) converges uniformly if and only if the ROC of the z transform X (z) of the sequence includes the unit circle. The ROC of X (z) consists of a ring in the z plane centered about the origin. That is the ROC of z transform of x (n) has values of z for which x(n) r^{-n} is absolutely summable.

$$X(z) = \sum_{n=0}^{\infty} |x(n)\, r^{-\mathrm{n}}|$$

- ROC does not contain any poles.
- If x (n) is a finite duration causal sequence or right sided sequence, then the ROC is entire z-plane except at z = 0.
- If x(n) is a finite duration anti-causal sequence or left sided sequence, then the ROC is entire z-plane except at z = ∞.
- If x (n) is an infinite duration causal sequence, ROC is exterior of the circle with radius a. i.e. |z| > a.
- If x (n) is an infinite duration anti-causal sequence, ROC is interior of the circle with radius a. i.e. |z| < a.
- If x (n) is a finite duration two sided sequence, then the ROC is entire z-plane except at z = 0 & z = ∞.

EXAMPLE (A): Determine the z transform of the following finite duration signals.

(a) x (n) = { 1, 2, 5, 4, 0, 1}
(b) x (n) = δ(n)
(c) x (n) = δ(n-k)
(d) Determine the z transform of

$$x(n) = \begin{cases} a^n, & 0 \le n \\ 0, & n<0 \end{cases}$$

SOLUTION:

(a) x (n) = { 1, 2, 5, 4, 0, 1}

Taking z transform, we get

$X(z) = 1 + 2z^{-1} + 5z^{-2} + 5z^{-3} + 4z^{-4} + z^{-6}$

(b) x (n) = δ(n)

X(z) = 1

(c) x (n) = δ(n-k)

x (n) = δ(n-k), k > 0

hence $X(z) = z^k$

ROC: Entire z plane except z = 0

d)

$$x(n) = \begin{cases} a^n, & 0 \le n \\ 0, & n<0 \end{cases}$$

The z transform of the given x(n) is

$$X(z) = Z[a^{\mathrm{n}}] = \sum_{n=-\infty}^{\infty} a^{\mathrm{n}}\, z^{-\mathrm{n}} = \sum_{n=0}^{\infty} a^{\mathrm{n}}\, z^{-\mathrm{n}}$$

We know that, $\sum a^n = \frac{1}{1-a}$ for a < 1

Hence,

$$X(z) = \frac{1}{1 - az^{-1}}$$

This converges when $|az^{-1}| < 1$ or $|z| > |a|$. Values for z for which X (z) = 0 are called of X (z), and values of z for which X (z) tends to infinity are called poles of X (z).

2.3 PROPERTIES OF Z TRANSFORM:

1) LINEARITY

The linearity property states that if

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z) \quad \text{and} \quad x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

Then

$$a_1 x_1(n) + a_2 x_2(n) \overset{z}{\longleftrightarrow} a_1 X_1(z) + a_2 X_2(z)$$

Where a_1 and a_2 are arbitrary constants. It implies that the z- transform of a linear combination of signals is the same linear combination of z transforms.

EXAMPLE (B): Determine the z transform of the signal

x (n) = δ(n+1) + 3 δ(n) + 6 δ(n-3) - δ(n-4)

SOLUTION:

From the linearity property, we have

X (z) = Z [δ (n+1)] + 3 Z [δ (n)] + 6 Z [δ (n-3)] – Z [δ (n-4)]

Using z transform pair, we obtain

x (n) = { 1, **3**, 6, 0, 0, -1}

2) TIME REVERSAL

The Time reversal property states that if

$$x(n) \overset{z}{\longleftrightarrow} X(z) \quad : \text{ROC } r_1 < |z| < r_2$$

Then

$$x(-n) \overset{z}{\longleftrightarrow} X(z^{-1}) \quad : \text{ROC } 1/r_1 < |z| < 1/r_2$$

EXAMPLE (C): Find the z transform of the signal x (n) = u (-n)

SOLUTION:

We know that

$$Z[u(n)] = \frac{z}{z-1} \qquad \text{ROC: } |z|>1$$

By using time reversal property, we obtain

$$Z[u(n)] = \frac{z^{-1}}{z^{-1}-1} \qquad \text{ROC: } |z|<1$$

3) TIME SHIFTING

The Time shifting property states that if

$$x(n) \overset{z}{\longleftrightarrow} X(z) \quad \text{Then} \quad x(n-k) \overset{z}{\longleftrightarrow} X(z)\, z^{-k}$$

The ROC of z^{-k} X (z) is the same as that of X(z) except for z = 0 if k > 0 and z = ∞ if k < 0.

EXAMPLE (D): Find the z transform of the signal $X(z)= z-1 /(1-3z^{-1})$

SOLUTION: $X(z)= z-1 /(1-3z^{-1}) = z^{-1} x_1(z)$

Where $X_1(z)= 1 /(1-3z^{-1})$

Here, from the time shifting property we have k=1 and $x(n) = (3)^n u(n)$

Hence $x(n)\ (3)^{n-1} u(n-1)$

4) SCALING

$$x(n) \overset{z}{\longleftrightarrow} X(z) \qquad : \text{ROC } r_1 < |z| < r_2$$

Then

$$a^n x(n) \overset{z}{\longleftrightarrow} X(a^{-1}z) \qquad : \text{ROC } |a|\, r_1 < |z| < |a|\, r_2$$

where a is arbitrary constant, which can be real or complex.

EXAMPLE (E): Find the z-transform of $x(n)=2^n u(n-2)$

SOLUTION: $x(n)=2^n u(n-2)$

$Z[u(n)] = 1/(1-z^{-1})$

$Z[u(n-2)] = z^{-2}/(1-z^{-1})$

$Z[2^n u(n-2)] = z^{-2}/(1-z^{-1})$ limit $z^{-1} \text{-->} 2z^{-1}$

$Z[2^n u(n-2)] = (2z^{-1})^2/(1-2z^{-1}) = (4z^{-2})/(1-2z^{-1})$

5) DIFFERENTIATION

$$x_1(n \overset{z}{\longleftrightarrow} X(z)$$

Then

$$n\, x(n) \overset{z}{\longleftrightarrow} -z\, dX(z)/dz$$

EXAMPLE (F): Find the z-transform of $x(n) = n^2 u(n)$

SOLUTION: $X(z) = Z[n^2 u(n)] = Z[n(n u(n)]$

$X(z) = z^{-1} \frac{d}{dz-1} [Z(nu(n))]$

$X(z) = z^{-1} \frac{d}{dz-1} \frac{z-1}{(1-z-1)2}$

$X(z) = z^{-1} [\frac{(1-z-1)2 - z-1[2(1-z-1)(-1)}{(1-z-1)4}]$

$X(z) = (1-z^{-1})(1-z^{-1}+2z^{-1}) / (1-z^{-1})^4$

$X(z) = \frac{1+z-1}{(1-z-1)3}$

6) CONVOLUTION

If

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z) \quad \text{and} \quad x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

Then

$$x_1(n) * x_2(n) \overset{z}{\longleftrightarrow} X_1(z) \cdot X_2(z)$$

EXAMPLE (G): Compute the convolution of the signals

$$x_1(n) = \{4, -2, 1\}$$

$$x_2(n) = \begin{cases} a^n, & 0 \le n \\ 0, & n<0 \end{cases}$$

SOLUTION: The z transform of the given signals are written as

$X_1(z) = 4 - 2z^{-1} + z^{-2}$

$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$

Therefore $X(z) = X_1(z). X_2(z) = 4 + 2z^{-1} + 3z^{-2} + 3z^{-3} + 3z^{-4} + 3z^{-5} - z^{-6} + z^{-7}$

Taking inverse z transform, we obtain

$x(n) = \{\mathbf{4}, 2, 3, 3, 3, 3, -1, 1\}$

7) CORRELATION PROPERTY

The Correlation of two sequences states that if

$$x_1(n) \overset{z}{\longleftrightarrow} X_1(z) \quad \text{and} \quad x_2(n) \overset{z}{\longleftrightarrow} X_2(z)$$

Then

$$\sum_{n=-\infty}^{\infty} x_1(l)\, x_2(-l) \overset{z}{\longleftrightarrow} X_1(z)\, X_2(z^{-1})$$

8) INITIAL VALUE THEOREM

If x(n) is a causal sequence with z transform X(z), the initial value can be determined by using the expression,

$$x(n) \overset{z}{\longleftrightarrow} X(z)$$

Then

$$x(0) \longleftrightarrow \lim_{n \to 0} x(n) \longleftrightarrow \lim_{z \to \infty} X(z)$$

9) FINAL VALUE THEOREM

If X (z) = Z [x (n)] and the poles of X (z) are all inside the unit circle, then the final value of the sequence, x(∞), can be determined by using the expression,

$$x(\infty) = \lim_{n \to \infty} x(n) = \lim_{z \to 1} (z-1)\, X(z)$$

EXAMPLE (H): If $X(z) = 2 + 3z^{-1} + 4z^{-2}$, find the initial value and final value of the corresponding sequence, x(n).

SOLUTION:

$$x(0) \longleftrightarrow \lim_{z \to \infty} [\ 2 + 3z^{-1} + 4z^{-2}\ ] = 2$$

$$x(\infty) = \lim_{z \to 1} [2 + z^{-1} + z^{-2} - 4z^{-3}] = 2 + 1 + 1 - 4 = 0$$

2.4 EVALUATION OF INVERSE Z TRANSFORM

The three basic methods of performing the inverse z transform, viz.

1. Partial fraction expansion Method
2. Power series expansion Method

1. PARTIAL FRACTION EXPANSION METHOD

In this method X (z) is first expanded into sum of simple partial fraction.

$$X(z) = \frac{a_0 z^m + a_1 z^{m-1} + \ldots\ldots + a_m}{b_0 z^n + b_1 z n^{n-1} + \ldots\ldots + b_n} \qquad \text{for } m \le n$$

First find the roots of the denominator polynomial

$$X(z) = \frac{a_0 z^m + a_1 z^{m-1} + \ldots\ldots + a_m}{}$$

$(z-p_1)(z-p_2)\ldots\ldots(z-p_n)$

2. POWER-SERIES EXPANSION METHOD

The z transform of a discrete time signal x (n) is given as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Expanding the above terms we have

$$x(z) = \ldots.. + x(-2) Z^2 + x(-1) Z + x(0) + x(1) Z^{-1} + x(2) Z^2 + \ldots..$$

This is the expansion of z transform in power series form. Thus sequence x(n) is given as

$$x(n) = \{ \ldots.. , x(-2), x(-1), x(0), x(1), x(2), \ldots\ldots\ldots\ldots. \}.$$

Power series can be obtained directly or by long division method.

2.5 ANALYSIS OF LINEAR TIME-INVARIANT SYSTEMS IN THE Z- DOMAIN

The z transform, plays an important role in the analysis and representation of discrete time LTI systems.
The convolution property of z transform states that,

y(n) = h(n) x(n)

$$y(n) = h(n) \otimes x(n) \longleftrightarrow Y(z) = H(z).X(z)$$

Where Y (z), H (z) and X (z) are the z transforms of system input, output and impulse response respectively.
H (z) is known as the transfer function of discrete time LTI systems. Sometimes, it is also called system function.

Important properties of the system:

1) Causality of discrete time LTI system:

The condition for causality of discrete time LTI system is that the impulse response of a causal discrete time LTI system is given as

h (n) = 0 for n < 0

This means that h (n) is right sided.

Also transfer function H (z) is the z transform of h (n). The ROC of H (z) of a causal discrete time system will be unity i.e., H (z) = 1

A discrete time LTI system is causal if and only if the ROC of its transfer function is the exterior of a circle, including infinity.

A discrete time LTI system which has a rational transfer function H (z) will be causal if and only if,

i) The ROC is exterior of a circle outside the outermost pole, and
ii) With H (z) expressed as a ratio of polynomials in z, the order of the numerator should be smaller than order of denominator.

3. Stability Criteria for a Discrete time LTI systems:

The stability of a discrete time LTI system is equivalent to its impulse response h (n) being absolutely summable, i.e.,

$$\sum_{-\infty}^{\infty} |h(k)| < \infty$$

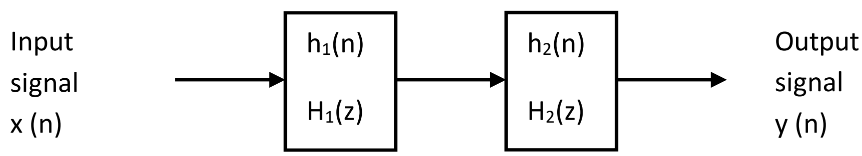
A discrete time LTI system is stable if and only if the ROC of its transfer function H (z) includes the unit circle z = 1.

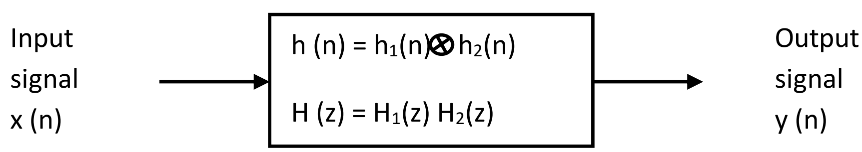
2.6 BLOCK DIAGRAM REPRESENTATION FOR DISCRETE TIME LTI SYSTEM

To analyze discrete time block diagrams such as series or cascade, parallel and feedback interconnections, the transfer function algebra if exactly the same as that for corresponding continuous time LTI system.

1. Series or Cascade inter connection:

The series connection of two discrete time LTI system is shown below:

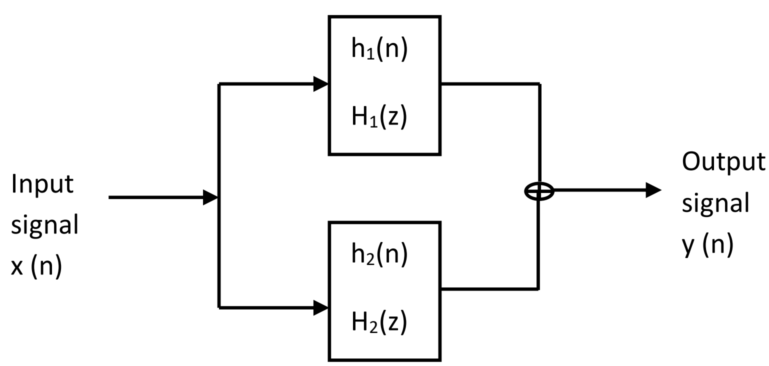




The overall impulse response of series connection of two discrete time systems can be determined by taking convolution sum between $h_1(n)$ and $h_2(n)$.Here, $h_1(n)$ and $h_2(n)$ are the impulse responses of two discrete time LTI systems.

2. Parallel interconnection:

The parallel connection of two discrete time LTI system is shown below:



The overall impulse response of parallel connection of two discrete time systems can be determined by addition of individual impulse responses of two discrete time LTI systems.