UNIT-I

Complex Analysis

Unit-01/Lecture-01

Introduction & Definition

Complex analysis, traditionally known as the **theory of functions of a complex variable**, is the branch of mathematical analysis that investigates functions of complex numbers. It is useful in many branches of mathematics, including algebraic geometry, number theory, applied mathematics; as well as in physics, including hydrodynamics, thermodynamics, nuclear, aerospace, mechanical and electrical engineering.

Murray R. Spiegel described complex analysis as "one of the most beautiful as well as useful branches of Mathematics".

Complex analysis is particularly concerned with analytic functions of complex variables (or, more generally, meromorphic functions). Because the separate real and imaginary parts of any analytic function must satisfy Laplace's equation, complex analysis is widely applicable to two-dimensional problems in physics.

Complex functions

A complex function is one in which the independent variable and the dependent variable are both complex numbers. More precisely, a complex function is a function whose domain and range are subsets of the complex plane.

For any complex function, both the independent variable and the dependent variable may be separated into real and imaginary parts:

$z = x + iy$ and
$w = f(z) = u(x,y) + iv(x,y)$
where $x, y \in \mathbb{R}$ and $u(x,y), v(x,y)$ are real-valued functions.

In other words, the components of the function *f*(*z*),

$u = u(x,y)$ and
$v = v(x,y),$

can be interpreted as real-valued functions of the two real variables, *x* and *y*.

The basic concepts of complex analysis are often introduced by extending the elementary real functions (e.g., exponential functions, logarithmic functions, and trigonometric functions) into the complex domain.

Analytic Function (Holomorphic functions)–

Given a complex-valued function *f* of a single complex variable, the **derivative** of *f* at a point z_0 in its domain is defined by the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

This is the same as the definition of the derivative for real functions, except that all of the quantities are complex. In particular, the limit is taken as the complex number *z* approaches z_0, and must have the same value for any sequence of complex values for *z* that approach z_0 on the complex plane. If the limit exists, we say that *f* is **complex-differentiable** at the point z_0. This concept of complex differentiability shares several properties with real differentiability: it is linear and obeys the product rule, quotient rule, and chain rule.

The necessary and sufficient condition for f(z) to be analytic

The relationship between real differentiability and complex differentiability is the following. If a complex function $f(x + i\,y) = u(x, y) + i\,v(x, y)$ is holomorphic, then *u* and *v* have first partial derivatives with respect to *x* and *y*, and satisfy the Cauchy–Riemann equations**:**

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

or, equivalently, the derivative of *f* with respect to the complex conjugate of *z* is zero:

$$\frac{\partial f}{\partial \overline{z}} = 0,$$

which is to say that, roughly, *f* is functionally independent from the complex conjugate of *z*.

If continuity is not a given, the converse is not necessarily true. A simple converse is that if *u* and *v* have *continuous* first partial derivatives and satisfy the Cauchy–Riemann equations, then *f* is holomorphic.

S.No.	Question	YEAR	MARKS
1	State and proof Cauchy-Riemann equation.	RGPV-June2009	7
2	Show that f(z)=$e^{-z^{-4}}$ and f(0)=0 is not analytic at z=0, although C-R equation are satisfied at this point.	RGPV-June2012	7

Unit-01/Lecture-02

Cauchy–Riemann Equations In Polar Form And Solved Questions

Cauchy–Riemann equations in polar form:[RGPV JUNE 2009](7)

In the system of coordinates given by the polar representation $z = re^{i\theta}$, the equations then take the form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

Theorem (Polar Form of the Cauchy-Riemann equations). [RGPV JUNE 2009]

Let $f(z) = f\left(r e^{i\theta}\right) = U(r,\theta) + i V(r,\theta)$

be a continuous function that is defined in some neighborhood of the point $z_0 = r_0 e^{i\theta_0}$. If all the partial derivatives

$U_r(r,\theta)$, $U_\theta(r,\theta)$, $V_r(r,\theta)$ and $V_\theta(r,\theta)$ are continuous at the point (r_0, θ_0),

and if the polar form of the Cauchy-Riemann equations,

$U_r(r_0,\theta_0) = \frac{1}{r_0} V_\theta(r_0,\theta_0)$ and $V_r(r_0,\theta_0) = \frac{-1}{r_0} U_\theta(r_0,\theta_0)$,

hold, then $f(z)$ is differentiable at $z_0 = r_0 e^{i\theta_0}$, and we can compute the derivative $f'(z_0)$ by using either

$$f'(z_0) = f'\left(r_0 e^{i\theta_0}\right) = e^{-i\theta_0}\left(U_r(r_0,\theta_0) + i V_r(r_0,\theta_0)\right), \quad \text{or}$$

$$f'(z_0) = f'\left(r_0 e^{i\theta_0}\right) = \frac{1}{r_0} e^{-i\theta_0}\left(V_\theta(r_0,\theta_0) - i U_\theta(r_0,\theta_0)\right).$$

Example. We know that $f(z) = z^2$ is differentiable and that $f'(z) = 2z$.

Furthermore, the polar coordinate form for $f(z)$ is

$$f(z) = f\left(r e^{i\theta}\right) = (r\cos\theta + i\, r\sin\theta)^2 = r^2\cos(2\theta) + i\, r^2\sin(2\theta).$$

Use the polar coordinate form of the Cauchy-Riemann equations and prove that $f(z)$ is differentiable for all $z \neq 0$.

Solution. It is easy to verify that polar form of the Cauchy-Riemann equations (3-22) are indeed satisfied for all $z \neq 0$.

$$U_r(r,\theta) = 2r\cos(2\theta) = \frac{1}{r}\,2r^2\cos(2\theta) = \frac{1}{r}V_\theta(r,\theta), \quad \text{and}$$

$$V_r(r,\theta) = 2r\sin(2\theta) = \frac{-1}{r}\,(-2r^2\sin(2\theta)) = \frac{-1}{r}U_\theta(r,\theta).$$

Moreover, the partial derivatives $U_r(r,\theta)$, $U_\theta(r,\theta)$, $V_r(r,\theta)$ and $V_\theta(r,\theta)$ are continuous for all $z \neq 0$.

By Theorem 3.5, $f(z) = r^2\cos(2\theta) + i\, r^2\sin(2\theta)$, is differentiable for all $z \neq 0$.

Therefore, using Equation (3-23) and (3-24), we have

$$\begin{aligned}
f'(z) &= f'\left(r e^{i\theta}\right) \\
&= e^{-i\theta}(U_r(r,\theta) + i\,V_r(r,\theta)) \\
&= e^{-i\theta}(2r\cos(2\theta) + i\,2r\sin(2\theta)) \\
&= 2r(\cos(\theta) + i\sin(\theta)) \\
&= 2z
\end{aligned}$$

and

$$\begin{aligned}
f'(z) &= f'\left(r e^{i\theta}\right) \\
&= \frac{1}{r}e^{-i\theta}(V_\theta(r,\theta) - i\,U_\theta(r,\theta)) \\
&= \frac{1}{r}e^{-i\theta}(2r^2\cos(2\theta) - i\,(-2r^2\sin(2\theta))) \\
&= 2r(\cos(\theta) + i\sin(\theta)) \\
&= 2z
\end{aligned}$$

as expected.

You might wonder why we required $z \neq 0$.

This happens because equations do not hold at $z \neq 0$.

Of course, for the function $f(z) = z^{\bar{z}}$, it is well known that $f'(0) = 0$.

Example: Prove that the function f(z)=z^2 is analytic function.

Sol. we have f(z)=z^2

We now separate real from imaginary parts. Letting x and y be real variables we have

$f(x+iy) = (x+iy)^2=x^2+2ixy-y^2=(x^2-y^2)+i(2xy)$.

Defining real functions u and v by f(x+iy)=u(x,y)+iv(x,y) and taking derivatives, we Have

$u(x,y)=x^2-y^2$ and $v(x,y)=2xy$

$$\frac{\partial u}{\partial x} = 2x,\ \frac{\partial u}{\partial y} = -2y,\ \ \frac{\partial v}{\partial x} = 2y,\ \frac{\partial v}{\partial y} = 2x.$$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ the Cauchy-Riemann relations are seen to be satisfied.

Example: Show that the function z|z| is not analytic anywhere.

Sol. We have w=z|z| =u+iv

And z=x+iy

Or $w=(x+iy)\sqrt{x^2 + y^2}$

Then, $u=x\sqrt{x^2 + y^2}$

$v=y\sqrt{x^2 + y^2}$

$$\frac{\partial u}{\partial x} = \frac{x^2+y^2\ +x^2}{\sqrt{x^2+y^2}},\ \frac{\partial u}{\partial y} = \frac{xy}{\sqrt{x^2+y^2}},\ \ \frac{\partial v}{\partial x} = \frac{xy}{\sqrt{x^2+y^2}},\ \frac{\partial v}{\partial y} = \frac{x^2+y^2\ +y^2}{\sqrt{x^2+y^2}}.$$

So that the C-R equations are not satisfied.

Example: Show that the function $f(z) = f(x+iy) = e^{-y}\cos(x) + ie^{-y}\sin(x)$

is differentiable and analytic for all $z = x + iy$ and find its derivative.

Sol. We first observe that

$$u(x,y) = e^{-y}\cos(x) \quad \text{and} \quad v(x,y) = e^{-y}\sin(x).$$

Then compute the partial derivatives and get

$$u_x(x,y) = -e^{-y}\sin(x) = v_y(x,y), \quad \text{and}$$

$$u_y(x,y) = -e^{-y}\cos(x) = -v_x(x,y).$$

This shows that C-R equation hold
Moreover, the partial derivatives $u_x(x,y)$, $u_y(x,y)$, $v_x(x,y)$ and $v_y(x,y)$ are continuous everywhere.

Example We know that $f(z) = z^2$ is differentiable and that $f'(z) = 2z$.

Furthermore, the Cartesian coordinate form for $f(z)$ is

$$f(z) = z^2 = (x + i y)^2 = (x^2 - y^2) + i(2xy).$$

Use the Cartesian coordinate form of the Cauchy-Riemann equations and find $f'(z)$.

Solution. It is easy to verify that Cauchy-Riemann equations are indeed satisfied:

$$u_x(x,y) = 2x = v_y(x,y) \quad \text{and} \quad u_y(x,y) = -2y = -v_x(x,y).$$

Now to compute $f'(z)$ gives,

$$f'(z) = u_x(x,y) + i v_x(x,y) = 2x + i2y = 2z,$$

and

$$f'(z) = v_y(x,y) - i u_y(x,y) = 2x - i(-2y) = 2z,$$

as expected.

Example 3.5. Show that $f(z) = \overline{z}$ is nowhere differentiable.

Solution. We have $f(z) = f(x + i y) = x - i y = u(x,y) + i v(x,y)$, where

$$u(x,y) = x \quad \text{and} \quad v(x,y) = -y.$$

Thus, for any point $z = x + i y$,

$$u_x(x,y) = 1 \quad \text{and} \quad v_y(x,y) = -1.$$

The Cauchy-Riemann equations are not satisfied at any point $z = x + i y$, so we conclude that

$f(z) = \bar{z}$ is nowhere differentiable, and not analytic.

S.No.	Question	YEAR	MARKS
1	Determine whether 1/z is analytic or not?	RGPV-June 2003	7
2	If f(z) is regular function of z prove that $(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})\ \lvert f(z) \rvert^2 = 4 \lvert f'(z) \rvert^2$.	RGPV-DEC.2014	7
3	Find Cauchy Riemann equation in polar form.	RGPV-JUNE.2014	3
4	Show that function f(z)=$\sqrt{\lvert xy \rvert}$ is not regular at z=0, although C-R equations are satisfied at this point.	RGPV-DEC.2013	7

Unit-01/Lecture-03

Theorem And Solved Questions

Theorem (Cauchy-Riemann conditions for differentiability). Assume that $f(z) = f(x + i y) = u(x, y) + i v(x, y)$

is a continuous function that is defined in some neighborhood of the point $z_0 = x_0 + i y_0$. If all the partial derivatives

$u_x(x, y)$, $u_y(x, y)$, $v_x(x, y)$ and $v_y(x, y)$ are continuous at the point (x_0, y_0) and if the Cauchy-Riemann equations

$u_x(x, y) = v_y(x, y)$ and $u_y(x, y) = -v_x(x, y)$

hold at $z_0 = x_0 + i y_0$, then $f(z)$ is differentiable at z_0, and the derivative $f'(z_0)$

$$f'(z_0) = f'(x_0 + i y_0) = u_x(x_0, y_0) + i v_x(x_0, y_0),$$

or

$$f'(z_0) = f'(x_0 + i y_0) = v_y(x_0, y_0) - i u_y(x_0, y_0).$$

Example. Prove that the function is analytic and find its derivative.

$$f(z) = u(x, y) + i v(x, y) = x^3 - 3 x y^2 + i(3 x^2 y - y^3).$$

Sol. We compute $u_x(x, y) = 3 x^2 - 3 y^2 = v_y(x, y)$ and $u_y(x, y) = -6 x y = -v_x(x, y)$, so the

Cauchy-Riemann Equations are satisfied. Moreover, the partial derivatives

$u_x(x, y)$, $u_y(x, y)$, $v_x(x, y)$ and $v_y(x, y)$ are continuous everywhere.

$f(z) = x^3 - 3 x y^2 + i(3 x^2 y - y^3)$ is differentiable everywhere, and, for derivative of f(z) ,

$$f'(z) = u_x(x, y) + i v_x(x, y)$$

$$= 3x^2 - 3y^2 + i\,6xy$$

$$= 3(x^2 - y^2 + 2ixy)$$

$$= 3(x + iy)^2$$

$$= 3z^2$$

Alternatively, solution

$$f'(z) = v_y(x, y) - i u_y(x, y)$$

$$= 3x^2 - 3y^2 - i(-6xy)$$

$$= 3(x^2 - y^2 + 2ixy)$$

$$= 3(x + iy)^2$$

$$= 3z^2$$

This result isn't surprising because $(x + iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$,

and so the function $f(z)$ is really our old friend $f(z) = z^3$.

Example Given
$f(z) = f(x + iy) = x^6 - 15x^4y^2 + 15x^2y^4 - y^6 + i(6x^5y - 20x^3y^3 + 6xy^5)$.

Show that this function is differentiable for all z, and find its derivative.

Sol. We compute the partial derivatives and get

$u_x(x, y) = 6x^5 - 60x^3y^2 + 30xy^4 = v_y(x, y)$, and

$u_y(x, y) = -30x^4y + 60x^2y^3 - 6y^5 = -v_x(x, y)$,

so that the Cauchy-Riemann Equations are satisfied. Moreover, the partial derivatives

$u_x(x, y)$, $u_y(x, y)$, $v_x(x, y)$ and $v_y(x, y)$ are continuous everywhere.

$f(z) = x^6 - 15x^4y^2 + 15x^2y^4 - y^6 + i(6x^5y - 20x^3y^3 + 6xy^5)$, is differentiable everywhere, and,

$$f'(z) = u_x(x, y) + i\, v_x(x, y)$$

$$= 6x^5 - 60x^3y^2 + 30xy^4 + i\,(30x^4y - 60x^2y^3 + 6y^5)$$

$$= 6\,(x^5 - 10x^3y^2 + 5xy^4 + i\,(5x^4y - 10x^2y^3 + y^5))$$

$$= 6\,(x + i\,y)^5$$

$$= 6z^5$$

Alternatively, solution

$$f'(z) = v_y(x, y) - i\, u_y(x, y)$$

$$= 6x^5 - 60x^3y^2 + 30xy^4 - i\,(-30x^4y + 60x^2y^3 - 6y^5)$$

$$= 6x^5 - 60x^3y^2 + 30xy^4 + i\,(30x^4y - 60x^2y^3 + 6y^5)$$

$$= 6\,(x^5 - 10x^3y^2 + 5xy^4 + i\,(5x^4y - 10x^2y^3 + y^5))$$

$$= 6\,(x + i\,y)^5$$

$$= 6z^5$$

This result isn't surprising because
$(x + i\,y)^6 = x^6 - 15x^4y^2 + 15x^2y^4 - y^6 + i\,(6x^5y - 20x^3y^3 + 6xy^5)$,

and so the function $f(z)$ is really our old friend $f(z) = z^6$.

Example. Show that the function $f(z) = f(x + i\,y) = e^{-y}\cos(x) + i\,e^{-y}\sin(x)$

is differentiable for all $z = x + i\,y$ and find its derivative.

Sol. . We first observe that

$u(x, y) = e^{-y}\cos(x)$ and $v(x, y) = e^{-y}\sin(x)$.

Then compute the partial derivatives and get

$$u_x(x, y) = -e^{-y}\sin(x) = v_y(x, y),$$ and

$$u_y(x, y) = -e^{-y}\cos(x) = -v_x(x, y).$$

Moreover, the partial derivatives $u_x(x, y)$, $u_y(x, y)$, $v_x(x, y)$ and $v_y(x, y)$ are continuous everywhere.
From Cauchy's theorem $f(z) = e^{-y}\cos(x) + i e^{-y}\sin(x)$, is differentiable everywhere.

Therefore, we have

$$f'(z) = u_x(x, y) + i v_x(x, y) = -e^{-y}\sin(x) + i e^{-y}\cos(x),$$ and

$$f'(z) = v_y(x, y) - i u_y(x, y) = -e^{-y}\sin(x) + i e^{-y}\cos(x).$$

Example. Show that the function $f(z) = f(x + i y) = e^{x^2-y^2}\cos(2xy) + i e^{x^2-y^2}\sin(2xy)$

is analytic for all $z = x + i y$ and find its derivative.

Sol. We first observe that

$$u(x, y) = e^{x^2-y^2}\cos(2xy)$$ and $$v(x, y) = e^{x^2-y^2}\sin(2xy).$$

Then compute the partial derivatives and get

$$u_x(x, y) = 2x e^{x^2-y^2}\cos(2xy) - 2y e^{x^2-y^2}\sin(2xy) = v_y(x, y),$$ and

$$u_y(x, y) = -2y e^{x^2-y^2}\cos(2xy) - 2x e^{x^2-y^2}\sin(2xy) = -v_x(x, y).$$

Moreover, the partial derivatives $u_x(x, y)$, $u_y(x, y)$, $v_x(x, y)$ and $v_y(x, y)$ are continuous everywhere.

By Cauchy's theorem $f(z) = e^{x^2-y^2}\cos(2xy) + i e^{x^2-y^2}\sin(2xy)$, is differentiable and analytic everywhere.

Therefore, we have

$$f'(z) = u_x(x, y) + i v_x(x, y)$$

$$= 2x e^{x^2-y^2}\cos(2xy) - 2y e^{x^2-y^2}\sin(2xy) + i\left(2y e^{x^2-y^2}\cos(2xy) + 2x e^{x^2-y^2}\sin(2xy)\right)$$

and

$f'(z) = v_y(x, y) - i u_y(x, y)$

$= 2x e^{x^2-y^2} \cos(2xy) - 2y e^{x^2-y^2} \sin(2xy) + i\left(2y e^{x^2-y^2} \cos(2xy) + 2x e^{x^2-y^2} \sin(2xy)\right)$

The Cauchy-Riemann conditions are particularly useful in determining the set of points for which a function f is differentiable.

Example Show that the function $f(z) = f(x + iy) = x^3 + 3xy^2 + i(3x^2 y + y^3)$

is differentiable at points that lie on the x-axis, and at points that lie on the y-axis, but $f(z)$ is nowhere analytic.

Sol. When we say a function is analytic at a point z_0 we mean that the function

is differentiable not only at z_0, but also at every point in some neighborhood of z_0. With this in mind, we proceed

to determine where the Cauchy-Riemann equations are satisfied. We write

$u(x, y) = x^3 + 3xy^2$ and $v(x, y) = 3x^2 y + y^3$,

and compute the partial derivatives:

$u_x(x, y) = 3x^2 + 3y^2$, $v_y(x, y) = 3x^2 + 3y^2$, and

$u_y(x, y) = 6xy$, $v_x(x, y) = 6xy$.

Here $u, v, u_x,$ and v_y are continuous, and

$u_x(x, y) = 3x^2 + 3y^2 = v_y(x, y)$

holds for all points (x, y) in the complex plane.

But $u_y(x, y) = -v_x(x, y)$ if and only if $6xy = -6xy$, which is equivalent to

$12xy = 0$.

Hence, the Cauchy-Riemann equations hold only at the points where $x = 0$ or $y = 0$.

According to Cauchy's theorem $f(z) = x^3 + 3xy^2 + i(3x^2 y + y^3)$ is differentiable only when $x = 0$ or $y = 0$,

which occurs only at points that lie on the coordinate axes. Furthermore, for any point on the coordinate axes, there contains an ϵ-neighborhood about it, in which there exist points where $f(z)$ is not differentiable.

we see that the function $f(z) = x^3 + 3xy^2 + i(3x^2y + y^3)$

is not analytic on either of the coordinate axes.

Therefore, $f(z) = x^3 + 3xy^2 + i(3x^2y + y^3)$ is nowhere analytic.

S.No.	**Questions**	**YEAR**	**MARKS**
1	Show that w=e^z is analytic function and determine f' (z).	RGPV DEC 2014	2
2	Show that the function e^x (cosy+isiny) is an analytic function and find its derivative.	RGPV JUNE 2008,2012	10

Unit-01/Lecture-04

Harmonic Conjugate

Definition (Harmonic Conjugate). If we have a function $u(x, y)$ that is harmonic on the domain D and if we can find another harmonic function $v(x, y)$ such that the partial derivatives for $u(x, y)$ and $v(x, y)$ satisfy the Cauchy-Riemann equations throughout D, then we say that $v(x, y)$ is a harmonic conjugate of $u(x, y)$. Furthermore, it then follows that the function $f(z) = f(x+iy) = u(x, y) + i v(x, y)$ is analytic on D.

The harmonic function is the real part of the given analytic function and the harmonic conjugate function
is the imaginary part of the given analytic function. .

Example Show that $u(x, y) = x^2 - y^2$ is a harmonic function and find a conjugate harmonic function $v(x, y)$, and an analytic function $f(z) = f(x + i y) = u(x, y) + i v(x, y)$.

Sol. Given $u(x, y) = x^2 - y^2$, we have $u_x(x, y) = 2x$ and $u_y(x, y) = 2y$ and the second partial

derivatives are $u_{xx}(x, y) = 2$ and $u_{yy}(x, y) = -2$. It follows that

$$u_{xx}(x, y) + u_{yy}(x, y) = 2 - 2 = 0,$$

hence $u(x, y) = x^2 - y^2$ is a harmonic function for all $z = x + i y$.

If we choose $v(x, y) = 2xy$, we have $v_x(x, y) = 2y$ and $v_y(x, y) = 2x$ and the second partial

derivatives are $v_{xx}(x, y) = 0$ and $v_{yy}(x, y) = 0$. It follows that

$$v_{xx}(x, y) + v_{yy}(x, y) = 0 - 0 = 0,$$

hence $v(x, y) = 2xy$ is a harmonic function for all $z = x + i y$.

Therefore, the harmonic conjugate of

$$u(x, y) = x^2 - y^2,$$

is

$$v(x, y) = 2xy.$$

Furthermore, u and v satisfy the Cauchy-Riemann equations

$u_x(x, y) = v_y(x, y) = 2x$, and

$u_y(x, y) = -v_x(x, y) = -2y$.

Therefore, $f(z) = f(x + i y) = x^2 - y^2 + i 2xy$ is an analytic function.

Alternative Solution.

The function $f(z) = z^2 = (x + i y)^2 = x^2 - y^2 + i 2xy$ is analytic for all values of z.

Hence, it follows from Theorem 3.8 that both

$u(x, y) = \mathrm{Re}[f(z)] = x^2 - y^2$, and

$v(x, y) = \mathrm{Im}[f(z)] = 2xy$,

are harmonic functions.

Example Show that $v(x, y) = 3x^2 y - y^3$ is a harmonic conjugate of $u(x, y) = x^3 - 3xy^2$.

Sol. Given $u(x, y) = x^3 - 3xy^2$, we have $u_x(x, y) = 3x^2 - 3y^2$ and $u_y(x, y) = -6xy$

and the second partial derivatives are $u_{xx}(x, y) = 6x$ and $u_{yy}(x, y) = -6x$. It follows that

$u_{xx}(x, y) + u_{yy}(x, y) = 6x - 6x = 0$,

hence $u(x, y) = x^3 - 3xy^2$ is a harmonic function for all $z = x + i y$.

Similarly, for $v(x, y) = 3x^2 y - y^3$, we have $v_x(x, y) = 6xy$ and $v_y(x, y) = 3x^2 - 3y^2$

and the second partial derivatives are $v_{xx}(x, y) = 6y$ and $v_{yy}(x, y) = -6y$. It follows that

$v_{xx}(x, y) + v_{yy}(x, y) = 6y - 6y = 0$,

hence $v(x, y) = 3x^2 y - y^3$ is a harmonic function for all $z = x + i y$.

Furthermore, u and v satisfy the Cauchy-Riemann equations

$u_x(x, y) = v_y(x, y) = 3x^2 - 3y^2$, and

$u_y(x, y) = -v_x(x, y) = -6xy$.

We see that $f(z) = f(x+iy) = (x^3 - 3xy^2) + i(3x^2y - y^3)$ is an analytic function.

Therefore, the harmonic conjugate of

$$u(x,y) = x^3 - 3xy^2,$$

is

$$v(x,y) = 3x^2y - y^3.$$

Alternative Solution.

The function $f(z) = z^3 = (x+iy)^3 = x^3 - 3xy^2 + i(3x^2y - y^3)$ is analytic for all values of z.

Hence, it follows that both

$$u(x,y) = \mathrm{Re}[f(z)] = x^3 - 3xy^2, \quad \text{and}$$

$$v(x,y) = \mathrm{Im}[f(z)] = 3x^2y - y^3,$$

are harmonic functions.

Therefore, the harmonic conjugate of

$$u(x,y) = x^3 - 3xy^2,$$

is

$$v(x,y) = 3x^2y - y^3.$$

S.No.	**Questions**	**YEAR**	**MARKS**
1	Find the imaginary part of the analytic function f(z) whose real part x^3-$3xy^2$+$3x^2$-$3y^2$.	RGPV DEC 2008	7
2	Show that the function e^x (xcosy-ysiny) is harmonic and find its conjugate.	RGPV FEB2010,DEC2013	7
3	Show that the function $\frac{1}{2}\log(x^2+y^2)$ is harmonic.	RGPV JUNE2014	2
4	Show that u=e^{-x}(xsiny-ycosy) is harmonic.	RGPV DEC2013	10
5	Show that the function u=e^{-2xy}sin(x^2-y^2) is harmonic. Find the conjugate function v and express u+iv as an analytic function of z.	RGPV JUNE2013	7

Unit-01/Lecture-05

Methods of Constructing an Analytic function

Method I: If u is given function then to find v

$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy = Mdx + Ndy$ [by C-R equations].....(1)

$$M = -\frac{\partial u}{\partial y} \; and \; N = \frac{\partial u}{\partial x}$$

$$\frac{\partial M}{\partial y} = -\frac{\partial^2 u}{\partial y^2} \; and \; \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x^2}$$

Since u is harmonic= $\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Hence equation (1) is exact DE.

So that dv can be integrated to get v.

Method II: Milne's Thomson Method

Type.(i) To construct analytic function f(z) in terms of z , when real part is given by the following formula:

$$f(z) = \int [\emptyset_1(z,0) - i\emptyset_2(z,0)]dz + C \dots\dots..(1)$$

Where $\emptyset_1(x,y) = \frac{\partial u}{\partial x}$, $\emptyset_2(x,y) = \frac{\partial u}{\partial y}$

Type.(ii) To construct analytic function f(z) in terms of z , when imaginary part v is given by the following formula:

$$f(z) = \int [\emptyset_1(z,0) + i\emptyset_2(z,0)]dz + C \dots\dots..(1)$$

Where $\emptyset_1(x,y) = \frac{\partial v}{\partial y}$ $\emptyset_2(x,y) = \frac{\partial v}{\partial x}$

Type.(iii) To construct analytic function f(z) in terms of z , where u-v is given :

Let U=u-v then

$$(1+i)f(z)=\int[\emptyset_1(z,0)-i\emptyset_2(z,0)]dz+C\ \ldots\ldots..(1)$$

Where $\emptyset_1(x,y)=\frac{\partial U}{\partial x}$, $\emptyset_2(x,y)=\frac{\partial U}{\partial y}$

Type.(iv) To construct analytic function f(z) in terms of z , where u+v is given :

Let V=u+v then

$$(1+i)f(z)=\int[\emptyset_1(z,0)+i\emptyset_2(z,0)]dz+C\ \ldots\ldots..(1)$$

Where $\emptyset_1(x,y)=\frac{\partial V}{\partial y}$, $\emptyset_2(x,y)=\frac{\partial V}{\partial x}$

Example Show that the function $u=x^3-3xy^2$ is harmonic and find the corresponding analytic function of this as the real part. **[RGPV Dec. 2011](7)**

Sol. Given $u(x,y)=x^3-3xy^2$

$\frac{\partial u}{\partial x}=3x^2-3y^2$ and $\frac{\partial^2 u}{\partial x^2}=6x$

$\frac{\partial u}{\partial y}=-6xy$ and $\frac{\partial^2 u}{\partial y^2}=-6x$

Hence $\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial x^2}=$ 0= u is harmonic.

Again, we have to find f (z) =u+iv.

By Milne's theorem, we have

$$f(z)=\int[\emptyset_1(z,0)+i\emptyset_2(z,0)]dz+C\ \ldots\ldots..(1)$$

Here $\emptyset_1(x,y)=\frac{\partial u}{\partial x}$, $\emptyset_2(x,y)=\frac{\partial v}{\partial y}$

We have $\emptyset_1(x,y)=\frac{\partial u}{\partial x}=3x^2-3y^2$

$$\emptyset_1(z,0)=3z^2$$

Again, we have

$\emptyset_2(x,y)=\frac{\partial v}{\partial y}=-\frac{\partial u}{\partial y}$ = 6xy

$$\emptyset_2(z,0)=0$$

Hence (1) becomes

$f(z) = \int [3z^2 + i0]dz + C = z^3 + iC$ **Ans.**

Theorem (Construction of a Conjugate). Let $u(x, y)$ be harmonic in an ϵ-neighborhood of the point (x_0, y_0).

Then there exists a conjugate harmonic function $v(x, y)$ defined in this neighborhood such that

$$f(z) = u(x, y) + i v(x, y),$$

is an analytic function.

Proof. A conjugate harmonic function $v(x, y)$ will satisfy the Cauchy-Riemann equations

$$u_x(x, y) = v_y(x, y) \quad \text{and} \quad u_y(x, y) = -v_x(x, y).$$

Assuming that such a function exists, we determine what it would have to look like by using a two-step process.

First, we integrate $v_y(x, y)$ (which should equal $u_x(x, y)$) with respect to y and get

$$v(x, y) = \int v_y(x, y)\, dy + C(x)$$

$$v(x, y) = \int u_x(x, y)\, dy + C(x)$$

where $C(x)$ is a function of x alone that is yet to be determined. Second, we compute $C'(x)$ by differentiating

both sides of this equation with respect to x and replacing $v_x(x, y)$ with $-u_y(x, y)$ on the left side, which gives

$$v_x(x, y) = \frac{d}{dx} \int u_x(x, y)\, dy + C'(x)$$

$$-u_y(x, y) = \frac{d}{dx} \int u_x(x, y)\, dy + C'(x)$$

It can be shown (we leave the details for the reader) that because u is harmonic, all terms except those involving x in the last equation

will cancel, revealing a formula for $C'(x)$ involving x alone. Elementary integration of the single-variable function $C'(x)$ can then be used to discover $C(x)$. We finally observe that the function $v(x,y)$ so created indeed has the properties we seek.

The functions $C(x)$ and $v(x,y)$ are computed with the formulas:

$$C(x) = \int \left(-u_y(x,y) - \frac{d}{dx}\left(\int u_x(x,y)\,dy\right)\right) dx,$$

and

$$v(x,y) = \int u_x(x,y)\,dy + C(x).$$

Example Show that $u(x,y) = xy^3 - x^3y$ is a harmonic function and find the harmonic conjugate $v(x,y)$.

Sol. We follow the construction process of Theorem. The first partial derivatives are

$$u_x(x,y) = y^3 - 3x^2y \quad \text{and} \quad u_y(x,y) = 3xy^2 - x^3.$$

To verify that $u(x,y)$ is harmonic, we compute the second partial derivatives and note that

$$u_{xx}(x,y) + u_{yy}(x,y) = -6xy + 6xy = 0,$$

so $u(x,y)$ satisfies Laplace's Equation

To construct $v(x,y)$, and

the Cauchy-Riemann equation $v_y(x,y) = u_x(x,y)$ and get

$$\begin{aligned} v(x,y) &= \int v_y(x,y)\,dy + C(x) \\ &= \int u_x(x,y)\,dy + C(x) \\ &= \int (y^3 - 3x^2y)\,dy + C(x) \\ &= \frac{1}{4}y^4 - \frac{3}{2}x^2y^2 + C(x) \end{aligned}$$

We now need to differentiate the left and right sides of this equation with respect to x,

$$v_x(x, y) = \frac{d}{dx}\left(\int u_x(x, y)\,dy\right) + C'(x).$$

Use the Cauchy-Riemann equation $-u_y(x, y) = v_x(x, y)$ to obtain

$$-3xy^2 + x^3 = 0 - 3xy^2 + C'(x)$$

It follows easily that

$$C'(x) = x^3,$$

then an easy integration yields $C(x) = \int x^3\,dx = \frac{1}{4}x^4 + c$, where c is a real constant.

For convenience, we can choose $c = 0$.

Therefore,

$$v(x, y) = \frac{1}{4}x^4 + \frac{1}{4}y^4 - \frac{3}{2}x^2y^2.$$

S.No.	Question	YEAR	MARKS
1	Show that the function $u=e^{-2xy}\sin(x^2-y^2)$ is harmonic. Find the conjugate function v and express u+iv as an analytic function of z.	**RGPV. JUNE 2012**	**7**
2	Show that the function $u=x^3-3xy^2$ is harmonic and find the corresponding analytic function of this as real part.	**RGPV. DEC 2011**	**7**
3	If $u=x^2-y^2$, find a corresponding analytic function by using Milne-Thomson method.	**RGPV. JUNE 2013**	**7**
4	If $u=x^3y-xy^3+2x^2-2y^2$ is the real part of an analytic function f(z)=u+iv, find v. Find also f(z) in terms of z.	**RGPV. JUNE 2010**	**10**

Unit-01/Lecture-06

Questions on Methods of Constructing an Analytic function

Example Given $f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$.

Show that this function is differentiable for all z, and find its derivative.

Sol. Recall the identities $x = \frac{z + \overline{z}}{2}$ and $y = \frac{z - \overline{z}}{2i}$ that were used in

They can be substituted in $f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$, and the result is

$$f(z) = g(z, \overline{z})$$

$$= \left(\frac{z + \overline{z}}{2}\right)^3 - 3\frac{z + \overline{z}}{2}\left(\frac{z - \overline{z}}{2i}\right)^2 + i\left(3\left(\frac{z + \overline{z}}{2}\right)^2\frac{z - \overline{z}}{2i} - \left(\frac{z - \overline{z}}{2i}\right)^3\right)$$

$$= \frac{z^3}{8} + \frac{3z^2\overline{z}}{8} + \frac{3z\overline{z}^2}{8} + \frac{\overline{z}^3}{8} + \frac{3z^3}{8} - \frac{3z^2\overline{z}}{8} - \frac{3z\overline{z}^2}{8} + \frac{3\overline{z}^3}{8} + i\left(-\frac{3iz^3}{8} - \frac{3iz^2\overline{z}}{8} + \frac{3iz\overline{z}^2}{8} + \frac{3i\overline{z}^3}{8} - \frac{iz^3}{8} + \frac{3iz^2\overline{z}}{8} - \frac{3iz\overline{z}^2}{8} + \frac{i\overline{z}^3}{8}\right)$$

$$= \frac{1}{2}z^3 + \frac{1}{2}\overline{z}^3 + i\left(-\frac{i}{2}z^3 + \frac{i}{2}\overline{z}^3\right)$$

$$= \frac{1}{2}z^3 + \frac{1}{2}\overline{z}^3 + \frac{1}{2}z^3 - \frac{1}{2}\overline{z}^3$$

$$= z^3$$

When we view $f(z) = g(z, \overline{z})$ as a function of the two variables z and $\overline{z}$, we see that

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}}(z^3 + 0\,\overline{z}^3) = 0.$$

Therefore, the complex form of the Cauchy-Riemann equations holds for all z and $f(z) = x^3 - 3xy^2 + i(3x^2y - y^3)$ is analytic for all z.

Indeed, $f(z) = z^3$ is the revealed formula of z alone, and we are permitted to use the rules for differentiation in Section 3.1, and we find that

$$f'(z) = 3z^2.$$

Example Given $f(z) = x^3 + 3xy^2 + i(3x^2y + y^3)$

is differentiable at points that lie on the x and y axes but $f(z)$ is nowhere analytic.

Sol. Recall the identities $x = \frac{z + \overline{z}}{2}$ and $y = \frac{z - \overline{z}}{2i}$ that were used in

They can be substituted in $f(z) = x^3 + 3xy^2 + i(3x^2y + y^3)$, and the result is

$$f(z) = g(z, \overline{z})$$

$$= \left(\frac{z + \overline{z}}{2}\right)^3 + 3\,\frac{z + \overline{z}}{2}\left(\frac{z - \overline{z}}{2i}\right)^2 + i\left(3\left(\frac{z + \overline{z}}{2}\right)^2 \frac{z - \overline{z}}{2i} + \left(\frac{z - \overline{z}}{2i}\right)^3\right)$$

$$= \frac{z^3}{8} + \frac{3z^2\overline{z}}{8} + \frac{3z\overline{z}^2}{8} + \frac{\overline{z}^3}{8} - \frac{3z^3}{8} + \frac{3z^2\overline{z}}{8} + \frac{3z\overline{z}^2}{8} - \frac{3\overline{z}^3}{8} + i\left(-\frac{3iz^3}{8} - \frac{3iz^2\overline{z}}{8} + \frac{3iz\overline{z}^2}{8} + \frac{iz^3}{8} - \frac{3iz^2\overline{z}}{8} + \frac{3iz\overline{z}^2}{8} + \frac{i\overline{z}^3}{4}\right)$$

$$= -\frac{1}{4}z^3 + \frac{3}{4}z^2\overline{z} + \frac{3}{4}z\overline{z}^2 - \frac{1}{4}\overline{z}^3 + i\left(-\frac{i}{4}z^3 - \frac{3}{4}iz^2\overline{z} + \frac{3}{4}iz\overline{z}^2 + \frac{i}{4}\overline{z}^3\right)$$

$$= -\frac{1}{4}z^3 + \frac{3}{4}z^2\overline{z} + \frac{3}{4}z\overline{z}^2 - \frac{1}{4}\overline{z}^3 + \frac{1}{4}z^3 + \frac{3}{4}z^2\overline{z} - \frac{3}{4}z\overline{z}^2 - \frac{1}{4}\overline{z}^3$$

$$= \frac{3}{2}z^2\overline{z} - \frac{1}{2}\overline{z}^3$$

When we view $f(z) = g(z, \overline{z})$ as a function of the two variables z and $\overline{z}$, we see that

$$\frac{\partial f}{\partial \overline{z}} = \frac{\partial}{\partial \overline{z}}\left(\frac{3}{2}z^2\overline{z} - \frac{1}{2}\overline{z}^3\right) = \frac{3}{2}z^2 - \frac{3}{2}\overline{z}^2 \neq 0.$$

Therefore, the complex form of the Cauchy-Riemann equations does not hold

and $f(z) = x^3 + 3xy^2 + i(3x^2y + y^3)$ is not analytic.

To determine where $f(z)$ has a derivative we must solve the equation
$\frac{3}{2}z^2 - \frac{3}{2}\overline{z}^2 = 0$.

First expand the quantity $\frac{3}{2} z^2 - \frac{3}{2} \overline{z}^2$ as follows.

$$\frac{3}{2} z^2 - \frac{3}{2} \overline{z}^2 = \frac{3}{2} (x + i y)^2 - \frac{3}{2} (x - i y)^2$$

$$= \frac{3}{2} (x^2 - y^2 + 2 i x y) - \frac{3}{2} (x^2 - y^2 - 2 i x y)$$

$$= \frac{3}{2} x^2 - \frac{3}{2} y^2 + 3 i x y - \frac{3}{2} x^2 + \frac{3}{2} y^2 + 3 i x y$$

$$= 6 i x y$$

Hence, the equivalent equation we need to solve is $6 i x y = 0$.

So we find that the complex form of the Cauchy-Riemann equations hold only when $x = 0$ or $y = 0$,

and according to Theorem, $f(z)$ is differentiable only at points that lie on the coordinate axes.

But this means that $f(z)$ is nowhere analytic because any ϵ-neighborhood about a point on either axis

contains points that are not on those axes.

Therefore $f(z) = x^3 + 3 x y^2 + i (3 x^2 y + y^3)$ is only differentiable at points on the x and y axe

S.No.	Question	YEAR	MARKS
1	Find the imaginary part of the analytic function whose real part is u=$\frac{sin2x}{cosh2y+cos2x}$.	**RGPV. JUNE 2003**	**7**
2	If u-v=(x-y)(x^2+4xy+y^2) and f(z)=u+iv is an analytic function of z=x+iy , find f(z) is terms of z.	**RGPV. JUNE 2006,2014**	**7**

Unit-01/Lecture-07

Line Integral

A **path of integration** is a parameterized plane curve $C = \{(x(t), y(t)),\ t \in [a, b]\}$,
where the functions $x(t)$ and $y(t)$ are continuous and have continuous first derivatives for $t \in [a, b]$. We have:

$$\forall t \in [a, b],\ z(t) = x(t) + iy(t) \quad \text{and} \quad dz = (x'(t) + iy'(t))\ dt$$

Let $f(z) = u(x, y) + iv(x, y)$ be a function continuous at every point of C.

Definition

$$\int_C f(z)\ dz = \int_a^b [(u(x(t), y(t)) + iv(x(t), y(t))](x'(t) + iy'(t))\ dt.$$

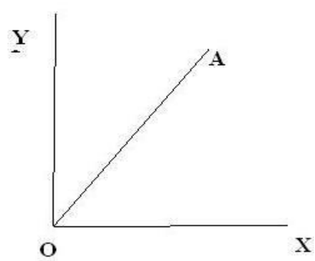
Example Find the value of the integral $\int_0^{1+i} (x - y + ix^2)dz$

(i) Along the straight line from z=0 to z=1+i,

(ii) Along the real axis from z=0 to z=1 and then along a line parallel to the imaginary axis from z=1 to z=1+i.

Sol. (i) Along the line OA : Equation of a straight line OA passing through (0,0) and (1,1) is,

y = x and z= x+iy =(1+i)x



dz =(1+i)dx

$\int_{OA} (x - y + ix^2)dz = \int_0^1 (x - x + ix^2)(1 + i)dx$

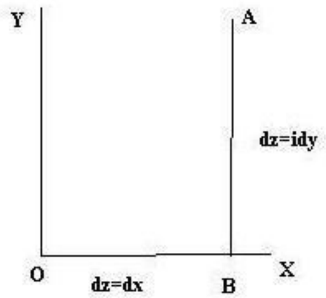
=(1+i) $\int_0^1 ix^2 dx$

= (1+i) i/3.

(ii)Along OB and then along BA

Along OB, from z=0 to z=1

Along BA from z=1 to z=1+i



Required integral,

= $\int_{OB}(x-y+ix^2)dz+\int_{BA}(x-y+ix^2)dz$ ......(1)

Now first integral = $\int_{OB}(x-y+ix^2)dz$

= $\int_0^1(x-0+ix^2)dx$=(3+2i)/6

Now second integral=$\int_{BA}(x-y+ix^2)dz$

= $\int_0^1(1-y+i)idy$

=(i-2)/2

Put these values in eq(1)

= (3+2i)/6+(i-2)/2

=5i/6-1/2. **Ans.**

Example: Evaluate $\int_0^{2+i}(\bar{z})^2dz$ **[RGPV JUNE 2005,2007,DEC2013](7)**

Along the real axis from z=0 to z=2 and then along a line parallel to the y axis from z=2 to z=2+i.

Sol. Since $(\bar{z})^2=(x-iy)^2=x^2-y^2-2ixy$

Along the line OBA ,where B is (2,0) and A is (2,1).

$I=\int_0^{2+i}(\bar{z})^2dz$

= $\int_{OB}(x^2-y^2-2ixy\ )dz+\int_{BA}(x^2-y^2-2ixy\ )dz$

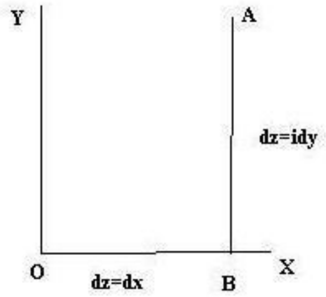
Now, Along OB and then along BA

Along OB, from y=0 , dz = dx and x varies from 0 to2.

$\int_{OB}(x^2-y^2-2ixy\ )dz$ = $\int_0^2(x^2)dx$=8/3

Along BA from x=2 , dz=i dy and y varies from 0 to 1,

$\int_{BA}(x^2-y^2-2ixy\ )dz$ = $\int_0^1(4-y^2-4iy)idy$=2+(11/3)i



Required integral,

$$= \int_{OB}(x^2 - y^2 \; - 2ixy \;)dz + \int_{BA}(x^2 - y^2 \; - 2ixy \;)dz$$

= 8/3+2+(11/3)i
=14/3+ (11/3) i. **Ans.**

Example Let C be given by $x(t) = \cos t, y(t) = \sin t$, where $t \in [0, \pi]$, i.e. C is the upper half unit-circle.

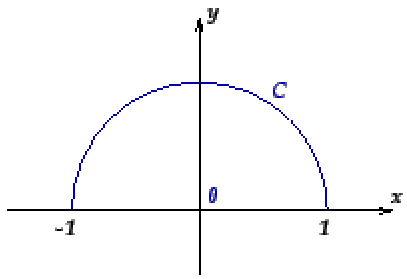


Figure 1:

Then $x'(t) = -\sin t$ and $y'(t) = \cos t$. we have:

$$\int_C z\, dz = \int_0^\pi (\cos t + i \sin t)(-\sin t + i \cos t)\; dt$$

$$= \int_0^\pi (-2 \sin t \cos t + i(\cos^2 t - \sin^2 t)\; dt$$

$$= \int_0^\pi (-\sin 2t + i \cos 2t)\; dt$$

$$= \left[\frac{1}{2}\cos 2t + \frac{i}{2}\sin 2t\right]_0^\pi = 0.$$

Ans.

Example Let us compute the integral of the function $z \mapsto \overline{z}$ along two different paths.

First let C be the same path as in previous example. If $z = \cos t + i \sin t$, then $\overline{z} = \cos t - i \sin t$. and we have:

$$\int_C \overline{z}\, dz = \int_0^{\pi} (\cos t - i \sin t)(-\sin t + i \cos t)\; dt$$

$$= \int_0^{\pi} i\; dt$$

$$= \pi i.$$

Ans.

Now we compute the integral of $z \mapsto \overline{z}$ along the segment from 1 to -1 on the x-axis; a parameterization of the segment is given by z=-t, for -1<t<1. Thus we have:

$$\int_C \overline{z}\, dz = \int_{-1}^{1} (-t)(-1)\; dt = \int_{-1}^{1} t\; dt = \left[\frac{1}{2}t^2\right]_{-1} 1 = 0.$$

Proposition

1. The integral is independent of the choice of the parameterization.
2. $\int_C (f+g)(z)\; dz = \int_C f(z)\; dz + \int_C g(z)\; dz$.
3. $\int_{C_1 \cup C_2} f(z)\; dz = \int_{C_1} f(z)\; dz + \int_{C_2} f(z)\; dz$, when C_1 and C_2 are two paths such that the endpoint of C_1 and the origin of C_2 are identical.
4. If C_1 and C_2 are opposite paths, $\int_{C_1} f(z)\; dz - \int_{C_2} f(z)\; dz$

S.NO.	QUESTION	YEAR	MARKS
1	Integrate z^2 along the straight line OA and also along the path OBA consisting of two straight line segments OB and BA where O is the origin , B is the point z=3 and A the point z=3+i.	RGPV DEC 2004	7
2	Evaluate $\int_{1-i}^{2+i} (2x + iy + 1)\; dz$, along the two paths: (i) X=t+1, y=$2t^2$-1 (ii) The straight line joining (1-i) and (2+i)	RGPV DEC 2008	7
3	Evaluate the integral $\int_c \frac{z^2-z+1}{z-1} dz$ where C is the circle |z|=1	RGPV JUNE2014	2
4	Prove that $\int_c \frac{1}{z-a} dz = 2\pi i$, where C is the circle |z-a|=r.	RGPV JUNE2012	10

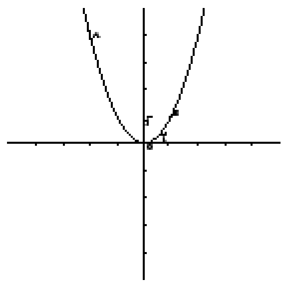
Unit-01/Lecture-08

Cauchy’s Theorem and Cauchy’ Integral Formula

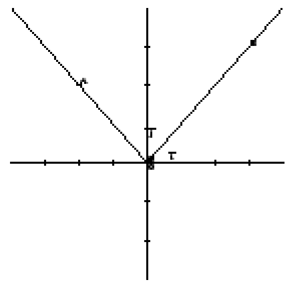
Cauchy's Theorems. [RGPV DEC 2002 AND 2011](7)

Let $C : [a,b] \longrightarrow \mathbb{C}$ be a path of integration. It is smooth at a point $t_0 \in [a,b]$ if C is derivable at t_0 and if its first derivative is continuous at t_0. The path is smooth if it is smooth at every point; it is smooth by parts if it is not smooth at only a finite number of points.

Example The arc AB of the parabola whose equation is y=x ^{2} is smooth.



(n) smooth



(o) non smooth at one point

Figure 2: Paths.

A **loop** is an integrating path whose origin and endpoints are identical. If it is smooth and does not intersect itself at another point, we will call it a **Jordan curve.** If we travel exactly once along the loop, we will call it a simple loop. The integral of a function f along a simple loop C will be denoted as follows:

$$\oint_C f(z)\, dz.$$

Note the little circle on the integration symbol.

A Jordan curve determines in the plane three disjoint regions: the curve itself, the interior (= a bounded region) and the exterior (= an unbounded region).

Cauchy's Integral Theorem or Cauchy’s Theorem: [RGPV Dec. 2002,2011,2013](7)

If f(z) is an analytic function and f’(z) is continuous at each point within and on a simple closed curve C, then $\int_C f(z)\, dz = 0$.

Note that the converse is not true: if f(z) is not analytic on the interior of C, the integral $\oint_C f(z)\, dz$ can either vanish or not. For example, compute the following integral:

$$I = \oint_{|z|=1} \frac{1}{z^2}\, dz.$$

Using the parameterization $z = \cos t + i\ \sin t,\ t \in [0, 2\pi]$, we can show that $I = 0$. The integral vanishes, despite the fact that the function fails to be analytic at 0, which is an interior point of the unit circle (defined here by the equation $|z| = 1$).

Example

1. $\int_C \sin z\, dz = 0$, for every Jordan curve C in the Cauchy-Argand plane.
2. Let $f(z) = \frac{e^z}{z^2-4}$ and let C be the unit circle. As f(z) is analytic on the closed unit disk, we have: $\int_C \frac{e^z}{z^2-4}\, dz = 0$.

Corollary *Let f(z) be a function defined and analytic on a connected domain* $\mathcal{D}$. *Let* C_1 *and* C_2 *be two paths with the same origin* A *and the same endpoint* B *such that both paths contain only interior points of* $\mathcal{D}$ *(see Figure 3). Then* $\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$.

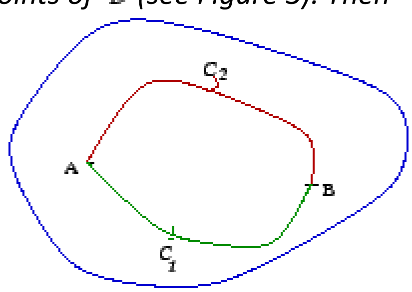


Figure 3: Two paths

Theorem *Consider two Jordan curves* C_1 *and* C_2 *such that all the points of* C_1 *are interior to* C_2 *(Figure 4). Let f(z) be a function, analytic on* C_1*, on* C_2 *and at every point of the ``annulus'' bounded by* C_1 *and* C_2. *Then:*

$$\int_{C_1} f(z)\, dz = \int_{C_2} f(z)\, dz$$

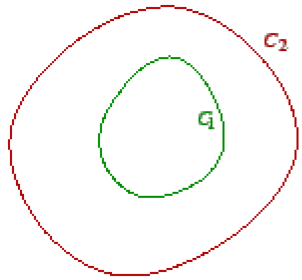


Figure 4: Loop within loop.

S.NO.	QUESTION	YEAR	MARKS
1	State and prove Cauchy theorem.	RGPV DEC 2011	7
2	State and prove Cauchy's integral theorem.	RGPV DEC 2013	10

Unit-01/Lecture-09

Cauchy' Integral Formula

Cauchy's Integral Formula. [RGPV DEC 2007, JUNE 2008](7)

Recall that if $z_0 \in \mathbb{C}$ and $R \geq 0$, $B(z_0, R) = \{z \in \mathbb{C}; |z - z_0| < R\}$ (see def open ball). As we work in the plane, it can happen that this open ball is also called an open disk.

Theorem *If f(z) is an analytic function within and on a closed curve C and 'a' is any point within C , then* $f(\alpha) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - \alpha}\, dz$

Where C *represents the circle whose centre is* α *and radius is equal to ρ.*

Another formulation of the above formula is as follows:

$$\int_C \frac{f(z)}{z - \alpha}\, dz = 2\pi i \cdot f(z)|_{z=\alpha}$$

Example

$$\int_{|z|=2} \frac{z^2}{z - i}\, dz = 2\pi i \cdot z^2|_{z=i} = 2\pi i \cdot (-1) = -2\pi i.$$

Ans.

Example We wish to compute the integral $I = \int_{|z|=2} \frac{1}{z^2+1}\, dz$.

The denominator vanishes at two points, i and $-i$, both inside the contour. We will decompose this contour into two Jordan curves by the following way: draw the diameter of the circle which coincides with the x-axis. Denote:

- C_1= the upper half circle together with the diameter, oriented positively.
- C_2= the lower half circle together with the diameter, oriented positively.

Then $I = \oint_{|z|=2} \frac{1}{z^2+1}\, dz = \oint_{C_1} \frac{1}{z^2+1}\, dz + \oint_{C_2} \frac{1}{z^2+1}\, dz$ (the variable z ``travels'' twice on the diameter, but in opposite directions).

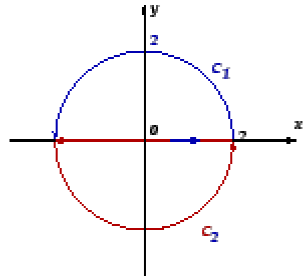


Figure 5:

Therefore we have:

$$= \int_{|z|=2} \frac{1}{z^2+1}\, dz$$

$$= \int_{|z|=2} = \frac{1}{(z-i)(z+i)}\, dz$$

$$= \int_{C_1} \frac{\frac{1}{z+i}}{z-i}\, dz + \int_{C_2} \frac{\frac{1}{z-i}}{z+i}\, dz$$

$$= 2\pi i \cdot \left[\frac{1}{z+i}\right]_{z=i} + 2\pi i \cdot \left[\frac{1}{z-i}\right]_{z=-i}$$

$$= 2\pi i \cdot \left(\frac{1}{2i} - \frac{1}{2i}\right) = 0.$$

Ans.

Question: Using Cauchy's integral formula, evaluate $\int_C \frac{e^{2z}}{(z+1)^4}\, dz$, where C is the circle |z|=3. **[RGPV JUNE 2011,2012,DEC 2014,DEC2013](7)**

Sol. By derivative of Cauchy integral formula:

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}}\, dz \qquad \text{........(1)}$$

Here, a=-1, n=3, and f(z)= e^{2z}

$$f'''(-1) = \frac{3!}{2\pi i} \int_C \frac{e^{2z}}{(z+1)^4}\, dz \qquad \text{.......(2)}$$

Since at pole z=a=-1 , so that |z|=|-1|=1<3 i.e. , point z=-1 lie in circle C.

f(z)=e^{2z} ,
f'(z)=$2e^{2z}$,
f''(z)=$4e^{2z}$,
f'''(z)=$8e^{2z}$,
f'''(-1)=$8e^{-2}$

Hence (2) becomes:

$$8e^{-2} = \frac{3}{\pi i} \int_C \frac{e^{2z}}{(z+1)^4}\, dz$$

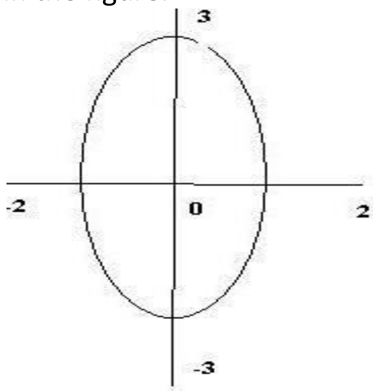
$\int_C \frac{e^{2z}}{(z+1)^4} dz\ e^{-2} = \frac{8\pi e^{-2}}{3} i$ **Ans.**

Question: If F(t) =$\int_C \frac{4z^2+z+5}{z-t} dz$, where c is the ellipse:**[RGPV JUNE 2009](7)**

$\frac{x^2}{4} + \frac{y^2}{9} = 1$. Find the values of (i)F(3.5) and (ii)F(i)

Sol. The give ellipse c is $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Here the point z=3.5 lies outside of the ellipse, while z=I lie inside the ellipse as shown in the figure.



(i) F (3.5) = $\int_C \frac{4z^2+z+5}{z-3.5} dz$,(1)

Since z=3.5 is not in c i.e. outside, by using Cauchy integral formula

$\int_C \frac{4z^2+z+5}{z-3.5} dz$=0, Hence (1) becomes F (3.5) =0.

Ans.

(ii) By Cauchy integral formula:

$\int_C \frac{f(z)}{z-t} dz = 2\pi i f(t),\ if\ t \epsilon$ C

$\int_C \frac{4z^2+z+5}{z-t} dz = 2\pi i[4z^2 + z + 5]$ z=t

Put t=i , in above equation

$\int_C \frac{4z^2+z+5}{z-i} dz = 2\pi i[-4 + i + 5]$ z=i

$\int_C \frac{4z^2+z+5}{z-i} dz = 2\pi[i - 1]$ z=i **Ans.**

S.NO	QUESTION	YEAR	MARKS
1	If $\int_C \frac{3z^2+7z+1}{z-i} dz$, where C is the circle $x^2+y^2=4$, find the values of f(3),f'(1-i) and f''(1-i).	RGPV DEC 2014	7
2	Evaluate$\int_C \frac{z^2-z+1}{z-1} dz$, where C is the circle |z|=1.	RGPV DEC 2014	2
3	Use Cauchy's integral formula, evaluate $\int_C \frac{e^{-2z}}{(z+1)^6} dz$,where C is |z|=2.	RGPV JUNE 2010	10

Unit-01/Lecture-10

Singularities

Some Definitions:

Zero of an analytic function: A zero of an analytic function f (z) is the value of z for which f (z) =0.

For Example: f(z)=(z+1)/z(z-2).

For zero of f(z), we put z+1=0, z=-1.

Order of zero: If f(z)=(z-a)$^m\varphi(z)$,

Such that $f(a) = 0, f'(a) = 0, f''(a) = 0, \ldots., f^{m-1}(a) = 0, f^m(a) \neq 0.$

Then, f (z) =has a zero of order m at z=a.

For example: f (z) = (z-i)^3/z^2+1.

For zero of f (z), put (z-i)^3=0, z=I of order m=3.

Singularity of an analytic function: A singular point of function f(z) is the point at which the function not analytic. In other words a point at which function f(z) is not defined.

For example: f(z)=1/(z+2i) has a singularity at z=-2i.

Isolated and Non-isolated singularity: A point z=a is said to be an isolated singularity of the function f(z) if f(z) is analytic at each point in some nbd of the point a defined by |z-a|<δ, except at the point a itself, i.e. if z=a is a singularity within a small circle of circumference with centre at the point a, then z=a is said to be an isolated singularity.

For example:

(i) Consider the function f(z)=(z+1)/z(z+2)

It is analytic everywhere except at z=0 and z=-2 . Thus z=0 and z=-2 are the only singularities of this function. There are no other singularities of f(z) in the nbd of z=0,-2. Hence z=0 and z=-2 are the isolated singularities of this function.

(ii) Let f(z)=Cot(π/z) = 1/tan(π/z)

It is not analytic at the points where tan(π/z)=0=tan(nπ) i.e, π/z=nπ

, z=1/n, where (n=1,2,3…..)

Thus, z=1, 1/2, 1/3,… are the isolated singularities of f(z) except z=0 because in the nbd of z=0 , there are infinite number of other singularities.

i.e., z=1/n if n is very large, z=0 is the non-isolated singularity of f (z).

Types of singularity:

Suppose f(z) is analytic within a region R except at z=a, which is an isolated singularity. Let C be a circle, whose center is such that 0<|z-a|< R.

Then by Laurent's series **,** $f(z) = \sum_{n=0}^{\infty} a_n \ (z-a)^n + \sum_{n=1}^{\infty} b_n \ (z-b)^{-n}$……(1)

The part

$\sum_{n=1}^{\infty} b_n \ (z-b)^{-n} \ \ldots \ldots (2)$, is called principal part of f(z), at z=a.

There are three distinct possibilities:

(i) Removable singularity:

All b_n's are zero i.e, no term in principal part of f(z).

i.e., $f(z) = \sum_{n=0}^{\infty} a_n \ (z-a)^n$, |z-a|< R

Then z=a is said to removable singularity of f(z).

Alternatively, $\log_{z\to 0} f(z)$, exist finitely, then z=a is a removable singularity.

For example: f(z)= sinz/z has removable singularity at z=0 since

Sinz/z= 1/z(z-z^2/3!+z^5/5!- …..)= 1-z^2/3! +z^4/5!-….

It has no term containing negative powers of z, i.e, principal part of f(z) has no term.

However, the singularity z=0 can be removed and the function be made analytic by defining sinz/z=1 at z=0.

(ii) Isolated essential singularity:

If the principal part of f(z) at z=a contains an infinite number of terms, then z=a is called an isolated essential singularity of f(z).

i.e., if there exists no finite value of m such that $\log_{z\to a}(z-a)^m f(z) = (finite\ non-zero\ constant)$,

then z=a is called an isolated essential singularity of f(z).

For example:

The function $e^{1/z}$ has essential singularity oat z=0, since

$e^{1/z}$=1+1/z+1/2!z^2+1/3!z^3+....

Has infinite number of terms in negative powers of z.

(iii) Pole: If the principal part of f(z) at z=a consists of a finite number of terms, say m, then the singularity at z=a is called a pole of order m of f(z).

A pole of order 1 is called a simple pole.

Alternatively, if $\log_{z\to a} f(z) = \infty$, then z=a is a pole of f(z).

Example: Find the kind of singularity of the function $\frac{cot\pi z}{(z-a)^2}$ at z=a and z=∞.**[RGPV DEC.2002]**

Sol. Given: $f(z) = \frac{cot\pi z}{(z-a)^2} = \frac{cos\pi z}{sin\pi z(z-a)^2}$

For the pole of f (z), taking denominator to zero.

I.e. sin(πz).$(z-a)^2$=0

= $(z-a)^2$=0 and sin (πz)=0

= z=a is a pole of order 2 and sin (πz)= sin (πn)

= z=n, (where n=0, 1,-1, 2,-2........), are simple poles.

If z=∞ is a limit of these poles then z = ∞ non-isolated essential singularity. **Ans**

Example: Find the kind of singularity of the function **[RGPV DEC. 2004]**

$$f(z) = \frac{(z-2)}{z^2}\sin(\frac{1}{z-1})$$

Sol. For poles of f(z) denominator of f(z) i.e. , z^2=0, i.e. , z=0 is a pole of order 2.

For zeros of f(z), the numerator of f(z)=0, i.e., $(z-2)\sin\left(\frac{1}{z-1}\right) = 0$

Z=2 and 1/(z-1)=nπ; i.e. z=2 , z=(1/nπ)+1. (Where n=1,-1,2,-2……)

Thus z=2 is a simple zero. The limit point of the zeros given by

z=(1/nπ)+1. (Where n=1,-1,2,-2……) [If n is large]

Hence z=1 is an isolated essential singularity. **Ans**

S.No.	Question	Year	Marks
1	Show that the function e^z has an isolated essential singularity at $z=\infty$.	RGPV DEC. 2003	7
2	Find the poles of the function $1/z^4+1$.	RGPV JUNE. 2003	7

Unit-01/Lecture-11

Residues Of F(z) At Pole

Residues of f(z) at pole:

Since analytic function f(z) can be expanded in a Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n\ (z-a)^n + \sum_{n=1}^{\infty} b_n\,(z-b)^{-n}$$

Then the coefficient of $\frac{1}{z-a}$, i.e., b_1 is called the residues of f(z) at pole z=a. It is denoted by

$[\text{Res } f(z)]_{z=a}$ or Res.f(a).

$[\text{Res } f(z)]_{z=a} = b_1 = \frac{1}{2\pi i}\int_C f(z)dz$, where C is the closed curve around the point z=a.

For example: $f(z)=\frac{sinz}{z^4} = \frac{1}{z^4}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} \ldots\ldots\right)$

$$= \left(\frac{1}{z^3} - \frac{1}{z\ 3!} + \frac{1}{5!}z - \frac{1^7}{7!}z^3 \ldots\ldots\right)$$

$[\text{Res } f(z)]_{z=0}$ =Coefficient of *(1/z-0)=-(1/3!)=(-1/6)*

Methods of finding out residues of f(z) at a Pole:

(i) Residue of f(z) at simple pole z=a

$[\text{Res } f(z)]_{z=a} = \lim_{z\to a}(z-a)f(z)$.

(ii) Residue of f(z) has pole of order m at z=a

$[\text{Res } f(z)]_{z=a,\ m} = \frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}(z-a)^m f(z)$.

(iii) Residue of f(z) at infinity

i.e. Residue of f(z) at $z=\infty=\lim_{z\to\infty}(-z)f(z)$

=-[Coefficient of 1/z in expansion of f(z) for values of z in the neighborhood by z=∞]

$=\frac{1}{2\pi i}\int_C f(z)dz,$

Where C is the closed contour enclosing all the singularities of f (z) except at infinity.

Example Find the order of each pole and residue at it of $\frac{\mathbf{1-2z}}{\mathbf{z(z-1)(z-2)}}$ **.[RGPV DEC 2001]**

Sol. Let f(z)= $\frac{\mathbf{1-2z}}{\mathbf{z(z-1)(z-2)}}$

For the poles of f(z): put z(z-1)(z-2)=0

Z=0, 1, 2 all are simple poles.

We know that **,**

$[\text{Res } f(z)]_{z=a} = \lim_{z\to a}(z-a)f(z)$.

(i) $[\text{Res } f(z)]_{z=0} = \lim_{z\to 0}(z-0)\frac{\mathbf{1-2z}}{\mathbf{z(z-1)(z-2)}} = \frac{1}{2}$.

(ii) $[\text{Res } f(z)]_{z=1} = \lim_{z\to 1}(z-1)\frac{\mathbf{1-2z}}{\mathbf{z(z-1)(z-2)}} = 1$

(iii) $[\text{Res } f(z)]_{z=2} = \lim_{z\to 2}(z-2)\frac{\mathbf{1-2z}}{\mathbf{z(z-1)(z-2)}} = \frac{3}{-2}$ **Ans.**

S.No	**Question**	**YEAR**	**MARKS**
1	Determine the pole of the function f (z) = $\frac{z^2}{(z-1)^2(z+2)}$ and the residue at each pole.	**RGPV JUNE 2002**	**7**
2	Evaluate the residues of $\frac{z^2-2Z}{(z+1)^2(z^2+1)}$ at each pole.	**RGPV JUNE 2010**	**10**
3	Find the residue of f (z) = $\frac{1-e^{2z}}{(z)^4}$ at its poles.	**RGPV JUNE 2013**	**7**
4	Determine the poles of the following function and residue at each pole: f(z)= $\frac{4-3z}{z(z-1)(z+2)}$	**RGPV JUNE 2012**	**10**

Unit-01/Lecture-12

Cauchy's Residue Theorem

Cauchy's Residue theorem:

Statement: If f(z) is an analytic function , except at a finite number of poles $a_{1,}$ $a_{2,}$ $a_{3,}$ $a_{4,....,}$ a_n with 9in a closed contour C and continuous on the boundary C, then

$\int_C f(z)dz$ = 2πi.∑ Res f(z) at the pole $a_{1,}$ $a_{2,}$ $a_{3,}$ $a_{4,....,}$ a_n in C

= 2πi(sum of residues at the poles within C).

Example: Evaluate $\int_C \frac{(4-3z)}{z(z-1)(z-2)}dz$, where C is the circle |z|=3/2.**[RGPV JUNE 2008,DEC 2013](7)**

Sol. Let f(z)= $\frac{(4-3z)}{z(z-1)(z-2)}$.

Using residue theorem:

$$\int_C f(z)dz = 2\pi\text{i}.\sum \text{ Res f(z)at each pole within C.}$$

For the poles of f(z): put z(z-1)(z-2)=0

= z=0,1,2 are simple poles.

At pole z=0 , then |z|=|0|=0<3/2, which lies inside the circle C.

At pole z=1, then |z|=|1|=1<3/2, which lies inside the circle C.

At pole z=2, then |z|=|2|=2>3/2, which lies outside the circle C.

We know that: $[\text{Res f (z)}]_{z=a} = \lim_{z\to a}(z-a)f(z)$.

Then

(iv) $[\text{Res f (z)}]_{z=0} = \lim_{z\to 0}(z-0)\frac{(4-3z)}{z(z-1)(z-2)} = \frac{4}{2} = 2$.

(v) $[\text{Res f (z)}]_{z=1} = \lim_{z\to 1}(z-1)\frac{(4-3z)}{z(z-1)(z-2)} = \frac{1}{-1} = -1$.

Hence (1) becomes: $\int_C \frac{(4-3z)}{z(z-1)(z-2)}dz$=2πi [2-1] =2πi. **Ans.**

Example: Evaluate $\int_C \frac{cos\pi z^2}{(z-1)(z-2)}dz$, where C is the circle |z|=3.**[RGPV JUE 2003,DEC.2011](7)**

Sol. Let f(z)= $\frac{cos\pi z^2}{(z-1)(z-2)}$.

Using residue theorem:

$$\int_C f(z)dz = 2\pi i. \sum \text{ Res f(z)at each pole within C.}$$

For the poles of f(z): put (z-1)(z-2)=0

= z=1, 2 are simple poles.

At pole z=1, then |z|=|1|=1<3, which lies inside the circle C.

At pole z=2, then |z|=|2|=2<3, which lies inside the circle C.

We know that: $[\text{Res f (z)}]_{z=a} = \lim_{z\to a}(z-a)f(z)$.

Then

(i) $[\text{Res f (z)}]_{z=1} = \lim_{z\to 1}(z-1)\frac{cos\pi z^2}{(z-1)(z-2)} = \frac{-1}{-1} = 1.$

(ii) $[\text{Res f (z)}]_{z=2} = \lim_{z\to 2}(z-2)\frac{cos\pi z^2}{(z-1)(z-2)} = \frac{1}{1} = 1$.

Hence (1) becomes: $\int_C \frac{cos\pi z^2}{(z-1)(z-2)}dz$=2πi [1+1] =4πi. **Ans.**

S.NO.	QUESTION	YEAR	MARKS
1	Solve $\int_c \frac{sin\pi z^2+cos\pi z^2}{(z-1)(z-2)}dz$, where C is the circle |z|=3.	RGPV DEC 2012,2013	7
2	Solve $\int_c \frac{cos\pi z^2}{(z-1)(z+1)}dz$, where C is the circle |z|=3.	RGPV DEC 2011	7
3	Evaluate $\int_C \frac{e^z}{(z-2)}dz$, where C is the circle (i) |z|=3 and (ii) |z|=1.	RGPV JUNE 2013	7

Unit-01/Lecture-13

Application Of Residues To Evaluate Real Integral

Application of Residues To Evaluate Real Integral:

Type I: Integral around unit circle

Consider the integral of the type

$$\int_0^{2\pi} f(cos\theta, sin\theta)\, d\theta$$

Where the integrand is a rational function of $sin\theta$& $cos\theta$, writing z=$e^{i\theta}$, so that, dz= i$e^{i\theta}$dθ

And $sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ & $cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

We have

$\int_0^{2\pi} f(cos\theta, sin\theta)\, d\theta$=$1/i \int_0^{2\pi} f(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right))\, dz/z$

Where C is the unit circle |z|=1 .Hence we can use residue theorem.

Example: Using contour integration evaluate the integral
[RGPV JUNE 2002,DEC.2003,JUNE2011](7)

$$\int_0^{2\pi} \frac{1}{2+cos\theta}\, d\theta$$

Sol. Let I =$\int_0^{2\pi} \frac{1}{2+cos\theta}\, d\theta$

Put z=$e^{i\theta}$, so that, dz= i$e^{i\theta}$dθ

dθ=dz/iz

Since $cos\theta = \frac{1}{2}\left(e^{i\theta} + \frac{1}{e^{i\theta}}\right) = \frac{1}{2}\left(z + \frac{1}{z}\right)$

Then we have,

I =$\int_0^{2\pi} \frac{1}{2+cos\theta}\, d\theta = \int_0^{2\pi} \frac{1}{2+\frac{1}{2}\left(z+\frac{1}{z}\right)}\, dz/iz = \int_0^{2\pi} \frac{2z}{4z+z^2+1}\, dz/iz$

Or I =2/i$\int_C \frac{1}{z^2+4z+1}\, dz$, where C is unit circle |z|=1

Let f(z)= $\frac{1}{z^2+4z+1}$

For poles: $z^2 + 4z + 1 = 0$, then the poles are z=-2±$\sqrt{3}$ are simple poles.

At pole z=-2+$\sqrt{3}$, then |z|=|-2+$\sqrt{3}$ |=0.267<1, which lies inside the circle C.

At pole z=-2-$\sqrt{3}$, then |z|=|-2-$\sqrt{3}$ 2|=2.73>1, which lies outside the circle C.

We know that: [Res f(z)]$_{z=a}$ = $\lim_{z\to a}(z-a)f(z)$.

Then

$$[\text{Res f(z)}]_{z=-2+\sqrt{3}} = \lim_{z=-2+\sqrt{3}}\left(z-(-2+\sqrt{3})\right)\frac{1}{z^2+4z+1} = \frac{1}{2\sqrt{3}}.$$

Hence: $\int_0^{2\pi}\frac{1}{2+cos\theta}d\theta$=2/i$\int_C\frac{1}{z^2+4z+1}dz$, =2πi [$\frac{1}{2\sqrt{3}}$] 2/i= $\frac{2\pi}{\sqrt{3}}$. **Ans.**

Type. II Evaluation of the integral of the type $\int_{-\infty}^{\infty} f(x)dx$

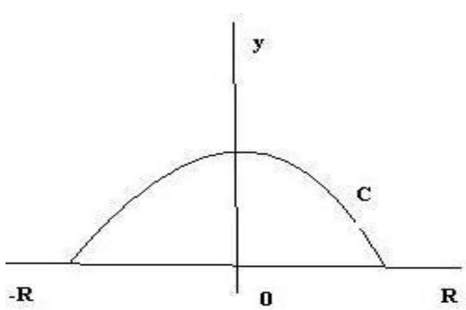
If f(z) is a function which is analytic in the upper half of the z-plane except at a finite number of poles in it , having no poles on the real axis an if further zf(z) tends to zero as |z|→ ∞, then by contour integration,

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum R^+$$

Where $\sum R^+$ represents the sum of the residues at the poles in the upper half of the plane.

Example: Evaluate by contour integration$\int_0^{\infty}\frac{1}{x^4+a^4}dx$, if a>0.

Sol. Let f(z)=$\frac{1}{z^4+a^4}$ and $\int_C f(z)dz = \int_C \frac{1}{z^4+a^4}dz$,



Where C is the contour consisting of

(i)Real axis from –R to R and

(ii)Semicircle C_R of radius R.

Therefore, $\int_C f(z)dz = \int_{-R}^{R} f(x)dx + \int_{C_R} f(z)dz$, For the poles of f(z) put $z^4+a^4=0$, $z^4=-a^4$ or z= $(-1)^4 a$

$$[(-1)^4=(\cos(2n+1)\pi+i\sin(2n+1)\pi)^{1/4}]$$

i.e. $z^4=(a)^4 e^{i(2n+1)\pi}$ or z= $a(e^{i(2n+1)\pi})^{1/4}$, where n=0, 1, 2, 3....

i.e. The poles are at z= $a(e^{i\pi})^{1/4}$, $a(e^{3\pi i})^{1/4}, a(e^{i5\pi})^{1/4}, a(e^{i7\pi})^{1/4}$

out of which first two lies in the upper half of the z-plane |z|=1.

Let α denotes any one of these poles then $\alpha^4=-(a)^4$

Now, $[\text{Res } f(z)]_{z=a}$ = $\lim_{z\to\alpha}(z-\alpha)f(z)$.

$$=\lim_{z\to\alpha}(z-\alpha)\frac{1}{z^4+a^4}.$$

$$=\lim_{z\to\alpha}\frac{1}{4z^3}=\frac{1}{4\alpha^3}=\frac{\alpha}{4\alpha^4}=-\frac{\alpha}{4a^4}$$

$[\text{Res } f(z)]_{z=a}e^{i\pi/4}$ = $-a\frac{e^{i\pi/4}}{4a^4}$ ......(i)

$[\text{Res } f(z)]_{z=a}e^{i\pi 3/4}$ = $-a\frac{e^{i\pi 3/4}}{4a^4}$ ....(ii)

Sum of the Residues = $\frac{-1}{4a^4}[ae^{\frac{i\pi}{4}}+ae^{\frac{i3\pi}{4}}]=\frac{-1}{4a^3}[e^{\frac{i\pi}{4}}+e^{\frac{i3\pi}{4}}]$

$$=\frac{-2i}{2i.4a^3}\left[e^{\frac{i\pi}{4}}-e^{\frac{-i\pi}{4}}\right]=\frac{-1}{4a^3}2i\sin\frac{\pi}{4}=\frac{-i\sqrt{2}}{4a^3}$$

Now $\left|\int_{C_R} f(z)dz\right|=\left|\int_{C_R}\frac{1}{z^4+a^4}dz\right|\le\int_{C_R}\frac{|dz|}{|z^4|-a^4}$

S.No.	Question	YEAR	MARKS
1	Apply calculus of residue to prove that $\int_c e^{\cos\theta}\cos(\sin\theta-n\theta)\,d\theta=\frac{2\pi}{n!}$, where n is positive integer.	RGPV DEC 2012	7
2	Define Residue and evaluate $\int_0^{2\pi}\frac{1}{1-2a\sin\theta+a^2}d\theta, 0<a<1$ by using residue theorem.	RGPV JUNE 2012	7
3	Evaluate $\int_0^{2\pi}\frac{\cos2\theta}{1-2a\cos\theta+a^2}d\theta, a^2<1$	RGPV FEB 2010	7
4	Evaluate the integral $\int_0^{\infty}\frac{\cos ax}{x^2+1}dx$.	RGPV JUNE 2014	7
5	Use calculus of residue to show that $\int_0^{2\pi}\frac{\cos2\theta}{5+4\cos\theta}d\theta=\frac{\pi}{6}$	RGPV JUNE 2013	7
6	Evaluate $\int_0^{2\pi}\frac{d\theta}{5+3\sin\theta}$.	RGPV DEC 2013	7
7	Apply the calculus of residue to show that $\int_0^{\pi}\frac{(1+2COS\theta)d\theta}{(5+3COS\theta)}$.	RGPV DEC 2013	7

8	Shoe that $\int_0^{2\pi} \frac{d\theta}{a+bcos\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$ $, a > b > 0.$	RGPV DEC 2013	10

REFERENCCE

BOOK	AUTHOR	PRIORITY
Engg Mathematics-III	B.S.Grewal	1
Engg Mathematics-III	Dr.D.C.Agarwal	2