UNIT - 2
Unit-02/Lecture-01
Solution of algebraic & transcendental equations by regula falsi method

ALGEBRAIC & TRANSCENDENTAL EQUATIONS

→ POLYNOMIAL -

An expression of the form

$P_n(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots a_{n-1} x + a_n$

where $a_0, a_1, \ldots\ldots a_n$ are constants, n is positive integer, is called a polynomial in x of degree n provided $a_0 \neq 0$

→ TRANSCENDENTAL EQUATION -

If a function $f(x)$ contains some other functions such as trigonometric, logarithmic, exponential etc. then $f(x) = 0$ is called transcendental equation.

Methods of Solving Equations -

i. Regula falsi method
ii. Newton - Raphson method
iii. Iterative method
iv. Secant method

1. Regula falsi Method -

The oldest method for computing the real root of a numerical equation is the method of false position, or "(Regula falsi)".

Let the root lie between a and b. These number a and b should be as close - together as possible. Since the root lies between a and b, the graph of $y = f(x)$ must cross the x-axis between $x = a$ and $x = b$ and $f(a)$ and $f(b)$ must have opposite signs.

$$x = \frac{a f(b) - b f(a)}{f(b) - f(a)} \quad \text{" } f(a).f(b) < 0 \text{ "}$$

[RGPV DEC(2013)] [7]

EXAMPLE -

Find a real root of the equation $x^3-2x-5=0$ by the method of false position correct to three decimal places

(RGPV, Bhopal, June 2010)

SOLUTION -

Let $f(x) = x^3 - 2x - 5$

$f(1) = 1 - 2 - 5 = -6$

$f(2) = 8 - 4 - 5 = -1 = -ve$

$f(3) = 27 - 6 - 5 = 16 = +ve$

$\Rightarrow f(2) \cdot f(3) < 0$

The root lies between 2 and 3

Let $a=2$ and $b=3$

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)} = \frac{2f(3) - 3f(2)}{f(3) - f(2)} = \frac{2(16) - 3(-1)}{16 - (-1)}$$

$$= \frac{32+3}{17} = 2.0588$$

Now, $f(2.0588) = (2.0588)^3 - 2(2.0588) - 5 = -0.3911$

and $f(3) = 16$, $f(2.0588) \cdot f(3) < 0$

$\therefore$ Root lies between 2.0588 and 3

$a = 2.0588$ and $b = 3$

$x_2 = 2.0813$

$f(2.0813) = (2.0813)^3 - 2(2.0813) - 5 = -0.1468$

and $f(3) = 16$, $f(2.0813) \cdot f(3) < 0$

$\therefore$ Root lies between 2.0813 and 3

$a = 2.0813$ and $b = 3$

$x_3 = 2.0897$

$f(2.0897) = (2.0897)^3 - 2(2.0897) - 5 = -0.0540$

and $f(3) = 16$, $f(2.0897) \times f(3) < 0$

$\therefore$ Root lies betwean 2.0897 and 3

$a = 2.0897$, $b = 3$

$x_4 = 2.0928$

$f(2.0928) = (2.0928)^3 - 2(2.0928) - 5 = 0.0195$

and $f(3) = 16$, $f(2.0928) \cdot f(3) < 0$

$\therefore$ Root lies between 2.0928 and 3

$a = 2.0928$ and $b = 3$

$x_5 = 2.0939$

$f(2.0939) = (2.0939)^3 - 2(2.0939) - 5 = -0.0073$

and $f(3) = 16$, $f(2.0939) \cdot f(3) < 0$

$\therefore$ Root lies between 2.0939 and 3

$a = 2.0939$, $b = 3$

$x_6 = 2.0943$

$f(2.0943) = (2.0943)^3 - 2(2.0943) - 5 = -0.0028$

and $f(3) = 16$, $f(2.0943) \cdot f(3) < 0$

$\therefore$ Root lies between 2.09~~39~~43 and 3

$a = 2.0943$ and $b = 3$

$x_7 = 2.0945$

Since $x_6 = x_7$ are correct upto three places of decimals so the required root of the given equation is 2.094. Ans.

EXAMPLE -

Find the real root of the equation $x \log_{10} x = 1.2$ by the method of false position (i.e. Regula falsi method) correct to four decimal places.

(RGPV, Bhopal, III Sem June 2009 Dec 2007, 2002)

SOLUTION -

Let $f(x) = x \log_{10} x - 1.2$

$f(2) = 2 \log_{10} 2 - 1.2 = -0.59794$;

$f(3) = 3 \log_{10} 3 - 1.2 = 0.23136$

$\therefore$ One root lies between 2 and 3

Taking $a = 2$, $b = 3$, $f(2) = -0.59794$, $f(3) = 0.23136$

By method of false Position, we have

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

$$x_1 = \frac{2f(3) - 3f(2)}{f(3) - f(2)} = \frac{2(0.23136) - 3(-0.59784)}{(0.23136) - (-0.59784)}$$

$$= 2.72102$$

$f(2.72102) = (2.72102)\log_{10} 2.72102 - 1.2 = -0.01709$

Since, $f(2.72102)$ and $f(3)$ are of opposite sign, so the root lies between 2.72102 and 3

$x_2 = 2.74021$

Now, $f(2.74021) = -0.00038$

Since $f(2.74021)\, f(3) < 0$, so the root lies between 2.74021 and 3

$x_3 = 2.74064$

Again $f(2.74064) = -0.00001$ and $f(3) = 0.23136$

Since $f(2.7406)\, f(3) < 0$, So the root lies between 2.7064 and 3

$x_4 = 2.74065$

Hence, the root Correct to four decimal places is 2.7407.

S.NO	RGPV QUESTIONS	Year	Marks
Q.1	Define algebraic and transcendental equations.	RGPV,DEC 2014	2
Q. 2	The equation $x^6 - x^4 - x^3 - 1 = 0$,has one real root between 1.4 and 1.5 . find the root to four decimal places by the method of false position.	RGPV,DEC 2014	3

Unit-02/Lecture-02

[RGPV JUNE(2014)] [2]

Newton Raphson Method

ii. NEWTON - RAPHSON METHOD

Let x_0 be an approximate root of $f(x)=0$ and let $x_1 = x_0 + h$ be the correct root so that $f(x_0+h)=0$

To find h, we expand $f(x_0+h)$ by taylor's Series

$$f(x_0+h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \ldots \quad [f(x_0+h)=0]$$

$$0 = f(x_0) + hf'(x_0)$$ [neglecting the secound and higher order derivative]

$$h = -\frac{f(x_0)}{f'(x_0)}$$

But $x_1 = x_0 + h$

Putting the value of h, we get

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

x_1 is better approximation than x_0.

$$\therefore x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

x_2 is better approximation than x_1.

Successive approximations are $x_3, x_4, \ldots\ldots x_{n+1}$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

which is the newton - Raphson formula.

NOTE 1. Newton method is the best known procedure for finding the roots of an equation. It is applicable to the solution of all types of equation i.e., algebraic and transcendental and also useful for calcutating complex roots.

2. This method is useful in case of large value of $f'(x)$. for large $f'(x)$, h will be small.

3. This formula converges rapidely. If the initial approximation x_0 is taken very close to the root α.

Thus proper choice of x_0 is very important for the success of this method.

→ CONVERGENCE OF NEWTON - RAPHSON FORMULA

By Newton - Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Let $\phi x = x - \frac{f(x)}{f'(x)} = \frac{xf'(x) - f(x)}{f'(x)}$

on differentiating both sides w.r.t 'x' we get

$$\phi'(x) = \frac{f'(x)[f'(x) + xf''(x) - f'(x)] - [xf'(x) - f(x)]f''(x)}{[f'(x)]^2}$$

$$\Rightarrow \phi'(x) = \frac{[f'(x)]^2 + xf'(x)f''(x) - [f'(x)]^2 - xf'(x)f''(x) + f(x)f''(x)}{[f'(x)]^2}$$

$$\Rightarrow \phi'(x) = \frac{f(x)\, f''(x)}{[f'(x)]^2}$$

For Convergence, $|\phi'(x)| < 1$

$$\frac{f(x) \cdot f''(x)}{[f'(x)]^2} < 1$$

$$f(x) f''(x) < [f'(x)]^2$$

EXAMPLE -

Use Newton - Raphson method to solve the equation $x^3 - 3x + 1 = 0$ correct to four decimal places

(RGPV., Bhopal, Dec 2009)

SOLUTION. $f(x) = x^3 - 3x + 1 = 0$, $f'(x) = 3x^2 - 3$

$f(0) = +1$, $f'(0) = -3$

$f(1) = 1 - 3 + 1 = -1$

Root lies between 0 and 1. Since $f(0) f(1) < 0$

Initial approximation is $x_0 = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0 - \frac{1}{-3} = \frac{1}{3} = 0.33333$$

$$f(0.33333) = (0.33333)^3 - 3(0.33333) + 1 = 0.03704 - 1 + 1 = 0.03704$$

$$f'(0.33333) = 3(0.33333)^2 - 3 = 0.33333 - 3 = -2.66667$$

$$x_2 = 0.33333 - \frac{f(0.33333)}{f'(0.33333)} = 0.33333 - \frac{0.03704}{-2.66667}$$

$$= 0.33333 + 0.01389 = 0.34722$$

$$f(0.34722) = (0.34722)^3 - 3(0.34722) + 1 = 0.04186 - 1.04166 + 1$$
$$= 0.0002$$

$$f'(0.34722) = 3(0.34722)^2 - 3 = 0.36169 - 3 = -2.63831$$

$$x_3 = 0.34722 - \frac{f(0.34722)}{f'(0.34722)} = 0.34722 - \frac{0.0002}{-2.63831}$$

$$= 0.34722 + 0.00008 = 0.3473$$

$$f(0.3473) = (0.3473)^3 - 3(0.3473) + 1 = 0.04189 - 1.0419 + 1 = 0$$

The root of the given equation is 0.3473

EXAMPLE - Using Newton's iterative Method, find the real root of $x\log_{10}x = 1.2$ correct to five decimal places.
(R.G.P.V, Bhopal, III Sem, feb 2010, Dec. 2005)

SOLUTION -

Let
$$f(x) = x\log_{10}x - 1.2$$
$$f(1) = -1.2 = -ve,$$
$$f(2) = 2\log_{10}2 - 1.2 = -0.59794 = -ve$$
$$f(3) = 3\log_{10}3 - 1.2 = +1.4314 - 1.2 = 0.23136 = +ve$$
$$f(2).f(3) < 0$$

So, a root of $f(x) = 0$ lies between 2 and 3

Let us take $x_0 = 2$

also, $f'(x) = \log_{10}x + x\cdot\frac{1}{x}\log_{10}e = \log_{10}x + 0.43429$

$\therefore$ Newton's iterative formula gives

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n \frac{x_n\log_{10}x_n - 1.2}{\log_{10}x_n + 0.43429}$$

$$= \frac{x_n\log_{10}x_n + 0.43429x_n - x_n\log_{10}x_n + 1.2}{\log_{10}x_n + 0.43429}$$

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[RGPV JUNE(2002)] [7]

$$= \frac{0.43429 x_n + 1.2}{\log_{10} x_n + 0.43429} \quad \ldots\ldots\ldots\ldots (1)$$

Putting $x_0 = 2$, the first approximation is

$$x_1 = \frac{0.43429 \times x_0 + 1.2}{\log_{10} x_0 + 0.43429} = \frac{0.43429 \times 2 + 1.2}{\log_{10} 2 + 0.43429}$$

$$= \frac{0.86858 + 1.2}{0.30103 + 0.43429} = 2.81$$

Similarly putting $n = 1, 2, 3, 4$ in (1), we get

$$x_2 = \frac{0.43429 \times 2.81 + 1.2}{\log_{10} 2.81 + 0.43429}$$

$$= 2.741$$

$$x_3 = \frac{0.43429 \times 2.741 + 1.2}{\log_{10} 2.741 + 0.43429}$$

$$= 2.74065$$

$$x_4 = \frac{0.43429 \times 2.74065 + 1.2}{\log_{10} 2.74065 + 0.43429}$$

$$= 2.74065$$

Clearly, $x_3 = x_4$

Hence, the required root is 2.74065 correct to five decimal places.

S.NO	RGPV QUESTIONS	Year	Marks
Q.1	Find the smallest positive root of the equation $x^3 - 2x + 0.5 = 0$ by Newton raphson method.	RGPV,DEC, 2014	2
Q. 2	Find the negative root of the equation $x^3 - 21x + 3500 = 0$ correct to 2 decimal places by newton raphson method.	RGPV,DEC 2013	7

3. ITERATION METHOD

Consider the equation $f(x) = 0$

We rewrite the equation in the form

$$x = \phi(x)$$

Let us draw two curves

$$y = x \text{ and } y = \phi(x)$$

The point of intersection of two curves is the root of (1)

Let $x = x_0$ be an initial approximate root, then first approximation x_1 is found by

$$x_1 = \phi(x_0)$$

Now taking x_1 as initial value, x_2 secound approximation is given by

$$x_2 = \phi(x_1) \text{ and so on.}$$

$$x_{n+1} = \phi(x_n)$$

This is also known as Sucessive approximation method.

NOTE :-

1. The rate of covergence is more if the value of $\phi'(x)$ is smaller
2. for real root, the method is very useful.

EXAMPLE -

Find a real root of $2x - \log_{10} x = 7$ correct to three decimal places using iteration method.
(R.G.P.V, Bhopal, II Sem, Dec 2006)

SOLUTION -

The given equation is

$$2x - \log_{10} x = 7$$

which can be written as

$$x = \frac{1}{2}(\log_{10} x + 7)$$

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[RGPV MARCH(2010),JUNE(2003,2008,2010),FEB(2005,2010)] [7]

Here, $\phi x = \frac{1}{2}(\log_{10} x + 7)$ ...... (1)

Let $x_0 = 3.8$

Putting the value of $x = 3.8$ in (1), we get

$$x_1 = \frac{1}{2}(\log_{10} 3.8 + 7)$$
$$= 3.79$$

On putting,

$x = 3.79$ in (1), we get

$$x_2 = \frac{1}{2}(\log_{10} 3.79 + 7)$$
$$= 3.7893$$

Again putting,

$x = 3.7893$ in (1), we get

$$x_3 = \frac{1}{2}(\log_{10} 3.7893 + 7)$$
$$= 3.7893$$

Since, $x_2 = x_3$

The root of the given equation is 3.7893.

S.NO	RGPV QUESTIONS	Year	Marks
Q.1	Find the root of the equation $3x = \cos x + 1$ by iterative method correct to 2 decimal places.	RGPV,JUNE 2014	3
Q. 2	Find the cube root of 15 correct to four significant figures by iterative method.	RGPV, DEC 2010	7

Unit-02/Lecture-04

[RGPV JUNE(2007)] [7]

Secant Method

4. SECANT METHOD (CHORD METHOD)

This method is quite similar to that of false position method and it is improved method over Regula falsi method. Here it is not necessary to fulfill the Condition $f(x_1)\,f(x_2) < 0$.

Here the graph of $f(x)$ is approximated by a Secant line (chord). The root may or may not lie in the interval $[a, b]$

apply the formula in the following way -

$$\frac{a f(b) - b f(a)}{f(b) - f(a)} = x_1 \qquad \text{I Time } \begin{bmatrix} a \\ b \end{bmatrix}$$

$$\text{II Time } \begin{bmatrix} b \\ x_1 \end{bmatrix}$$

$$\frac{b f(x_1) - x_1 f(b)}{f(x_1) - f(b)} = x_2 \qquad \text{III Time } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\text{IV Time } \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$$

$$\frac{x_1 f(x_2) - x_2 f(x_1)}{f(x_2) - f(x_1)} = x_3 \qquad \text{V Time } \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

and So on....

If $f(x_{n-1}) = f(x_n)$ then the method fails. But the rate of Convergence in secant method is greater than that of Regula falsi.

EXAMPLE -

Given the equation $x^4 - x - 10 = 0$, determine the initial approximations for finding its smallest positive root. Use these to find the root correct to three decimal places with Secant method.

SOLUTION -

Let $f(x) = x^4 - x - 10 = 0$

$f(1) = 1 - 1 - 10 = -10$, $a = 1$

$f(2) = 16 - 2 - 10 = +4$, $b = 2$

By Secant Method,

$$x_1 = \frac{a f(b) - b f(a)}{f(b) - f(a)}$$

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1.71429 \\ 1.83853 \end{bmatrix}$$

$$1.85500 \begin{bmatrix} 1.85778 \\ 1.85563 \end{bmatrix}$$

$$x_1 = \frac{1f(2) - 2f(1)}{f(2) - f(1)} = \frac{1(4) - 2(-10)}{4 - (-10)} = \frac{24}{14} = \frac{12}{7} = 1.71429$$

$$f(1.71429) = (1.71429)^4 - (1.71429) - 10$$
$$= 8.63649 - 1.71429 - 10 = -3.0778$$

Taking $a = 2$, $b = 1.71429$

$$x_2 = \frac{1.71429 f(2) - 2f(1.71429)}{f(2) - f(1.71429)}$$

$$= \frac{1.71429(4) - 2(-3.0778)}{(4) - (-3.0778)} = \frac{13.01276}{7.0778} = 1.83853$$

$$f(1.83853) = (1.83853)^4 - (1.83853) - 10 = -0.41283$$

Taking $a = 1.71429$ $b = 1.83853$.

$$x_3 = \frac{1.83853 f(1.71429) - 1.71429 f(1.83853)}{f(1.71429) - f(1.83853)}$$

$$= \frac{1.83853(-3.0778) - 1.71429(-0.41283)}{-3.0778 - (-0.41283)}$$

$$= \frac{-4.95092}{-2.66497} = 1.85778$$

$$f(1.85778) = (1.85778)^4 - (1.85778) - 10 = 0.05401$$

Taking $a = 1.83853$, $b = 1.85778$

$$x_4 = \frac{1.85778 f(1.83853) - 1.83853 f(1.85778)}{f(1.83853) - f(1.85778)}$$

$$= \frac{1.85778(-0.41283) - 1.83853(0.05401)}{(-0.41283) - 0.05401}$$

$$= \frac{-0.86625}{-0.46684} = 1.85561$$

$$f(1.85561) = (1.85561)^4 - (1.85561) - 10 = 0.00062$$

Taking $a = 1.85778$ and $b = 1.85561$

$$x_5 = \frac{1.85561\, f(1.85778) - 1.85778\, f(1.85561)}{f(1.85778) - f(1.85561)}$$

$$= \frac{1.85561(0.05401) - 1.85778(0.00062)}{0.05401 - 0.00062}$$

$$= \frac{0.099064}{0.05339} = 1.85558$$

$$x_4 = x_5$$

Hence, the root of the given equation is 1.8556.

Unit-02/Lecture-05

[RGPV DEC(1999)] [7]

Solution of simultaneous linear equations by gauss elimination method

SOLUTION OF SIMULTANEOUS LINEAR EQUATION

1. DIRECT METHOD
   a) Gauss elimination method
   b) Gauss - Jordan method
   c) Crouts method (factorisation method)

2. ITERATIVE METHOD
   a) Jacobi method
   b) Gauss - seidel method

1.a) Gauss Elimination Method :-

In this method the unknowns of equation below are eliminated and the system is reduced to an upper triangular system. the unknowns are obtained by back substitution.

EXAMPLE - Solve by Gauss elimination method :

$$2x-y+3z=9$$
$$x+y+z=6$$
$$x-y+z=2$$

(R.G.P.V Bhopal, June 2009)

SOLUTION - Here we have

$$2x-y+3z=9 \quad \text{----} (1)$$
$$x+y+z=6 \quad \text{----} (2)$$
$$x-y+z=2 \quad \text{-----} (3)$$

Step 1 · Interchanging (1) and (2) we have.

$$x+y+z=6 \quad \text{----} (1)$$
$$2x-y+3z=9 \quad \text{-----} (2)$$
$$x-y+z=2 \quad \text{-----} (3)$$

Step 2. Substracting 2(1) from (2) and (1) from (3) we get.

$$-3y + z = 3$$

$$-2y = -4 \Rightarrow y = 2$$

step 3. Backward Substitution

Putting the value of $y = 2$ in (4), we have

$$-6 + z = -3 \Rightarrow z = 3$$

Putting the values of y and z in (1) we get

$$x + 2 + 3 = 6 \Rightarrow x = 1$$

Hence $x = 1$, $y = 2$ and $z = 3$

S.NO	RGPV QUESTIONS	Year	Marks
Q.1	Solve by gauss elimination method the following system of equations: 10x+y+2z=13 ,3x+10y+z=14 , 2x+3y+10z=15.	RGPV JUNE 2013	7

[RGPV DEC(2001)] [7]

Gauss Jordan Method

1.b) GAUSS - JORDAN METHOD -

This is modification of Gauss elimination method. By this method we eliminate unknowns not only from the equations below but also from the equation above. In this way the system is reduced to a diagonal matrix. Finally each equation consists of only one unknown and thus, we get the solution. Here, the labour of backward substitution for finding the unknown is saved.

EXAMPLE: Apply Gauss-Jordan method to solve the equations:

$$x+y+z=9$$
$$2x-3y+4z=13$$
$$3x+4y+5z=40$$

(R.G.P.V Bhopal III Sem, Dec 2007)

SOLUTION:

The following system of linear equation is written in matrix form:

$$AX = B$$

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \\ 3 & 4 & 5 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 9 \\ 13 \\ 40 \end{bmatrix}$.

By using Gauss-Jordan method we have

$$\begin{bmatrix}1&1&1\\2&-3&4\\3&4&5\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}9\\13\\40\end{bmatrix} \Rightarrow \begin{bmatrix}1&1&1\\0&-5&2\\0&1&2\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}9\\-5\\13\end{bmatrix} \begin{matrix}R_2\to R_2-2R_1\\R_3\to R_3-3R_1\end{matrix}$$

$$\Rightarrow \begin{bmatrix}1&1&1\\0&-5&2\\0&0&\frac{12}{5}\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}9\\-5\\12\end{bmatrix} R_3\to R_3+\frac{1}{5}R_2$$

$$\Rightarrow \begin{bmatrix}1&0&\frac{7}{5}\\0&-5&2\\0&0&\frac{12}{5}\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}8\\-5\\12\end{bmatrix} R_1\to R_1+\frac{1}{5}R_2$$

$$\Rightarrow \begin{bmatrix}1&0&0\\0&-5&0\\0&0&\frac{12}{5}\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\-15\\12\end{bmatrix} \begin{matrix}R_1\to R_1-\frac{7}{12}R_3\\R_2\to R_2-\frac{5}{6}R_3\end{matrix}$$

$$\Rightarrow \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}1\\3\\5\end{bmatrix} \begin{matrix}R_2\to \frac{1}{5}R_2\\R_3\to \frac{5}{12}R_3\end{matrix}$$

Hence, $x=1$, $y=3$, $z=5$

S.NO	RGPV QUESTIONS	Year	Marks
Q.1	Apply gauss Jordan method to find the solution of the following system of equation: 10x+y+z=12 , 2x+10y+z=13 , x+y+5z=7	RGPV DEC 2014	7

Unit-02/Lecture-07

Crouts method

1.C > CROUTS - TRIANGULARISATION METHOD -

This method is also called as decomposition method. the coefficents matrix A of the system $AX=B$ is decomposed or factorised as the product of lower triangular matrix and an upper triangular matrix U. this method is based on the fact that every square matrix A. is the product of a lower triangular matrix and an upper triangular matrix.

EXAMPLE - Solve the equations:

$$3x+y+z=4$$
$$x+2y+2z=3$$
$$2x+y+3z=4$$

by Crout's method (RGPV Bhopal, Feb. 2010)

SOLUTION : The given equation are written in matrix form:

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$$A = LU$$

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 2 & 2 \\ 2 & 1 & 3 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & 33 \end{bmatrix} \begin{bmatrix} 1 & u_{12} & u_{13} \\ 0 & 1 & u_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} l_{11} & l_{11}u_{12} & l_{11}u_{13} \\ l_{21} & l_{21}u_{12}+l_{22} & l_{21}u_{13}+l_{22}u_{23} \\ l_{31} & l_{31}u_{12}+l_{32} & l_{31}u_{13}+l_{32}u_{23}+l_{33} \end{bmatrix}$$

Equating, we get

$l_{11} = 3$ $\quad l_{11}u_{12} = 1$ $\quad l_{11}u_{13} = 1$

$l_{21} = 1$ $\quad l_{21}u_{12}+l_{32} = 2$ $\quad l_{21}u_{13}+l_{22}u_{23} = 2$

$l_{31} = 2$ $\quad l_{31}u_{12}+l_{32} = 1$ $\quad l_{31}u_{13}+l_{32}u_{23}+l_{33} = 3$

From the above, we get

$l_{11} = 3$ $\quad u_{12} = \frac{1}{3}$ $\quad u_{13} = \frac{1}{3}$

$l_{21} = 3$ $\quad l_{22} = \frac{5}{3}$ $\quad u_{23} = 1$

$l_{31} = 2$ $\quad l_{32} = \frac{1}{3}$ $\quad l_{33} = 2$

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$$A = LU = \begin{bmatrix} 3 & 0 & 0 \\ 3 & 5/3 & 0 \\ 2 & 1/3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1/3 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

The given equations are

$$AX = B$$

$$LUX = B \quad \text{------} \quad (1)$$

Let $UX = Y$ so that (1) becomes

$$LY = B$$

$$\begin{bmatrix} 3 & 0 & 0 \\ 1 & 5/3 & 0 \\ 2 & 1/3 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix}$$

$3y_1 = 4 \quad \Rightarrow \quad y_1 = 4/3$

$y_1 + 5/3\, y_2 = 3 \quad \Rightarrow \quad 4/3 + 5/3\, y_2 = 3 \quad \Rightarrow \quad y_2 = 1$

$2y_1 + 1/3\, y_2 + 2y_3 = 4 \quad \Rightarrow \quad 2(4/3) + 1/3(1) + 2y_3 = 4$

$\Rightarrow 2y_3 = 1 \quad \Rightarrow \quad y_3 = \frac{1}{2}$

we know $UX = Y$

$$\begin{bmatrix} 1 & 1/3 & 1/3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4/3 \\ 1 \\ 1/2 \end{bmatrix}$$

$$x + \frac{y}{3} + \frac{z}{3} = \frac{4}{3}$$

$$y + z = 1 \quad \Rightarrow \quad z = \frac{1}{2}$$

Backward Substitution

$$y + \frac{1}{2} = 1 \quad \Rightarrow \quad y = \frac{1}{2}$$

$$x + \frac{1}{3}\left(\frac{1}{2}\right) + \frac{1}{3}\left(\frac{1}{2}\right) = \frac{4}{3} \quad \Rightarrow \quad x + \frac{1}{3} = \frac{4}{3} \quad \Rightarrow \quad x = 1$$

$$x = 1, \quad y = \frac{1}{2}, \quad z = \frac{1}{2}$$

S.NO	RGPV QUESTIONS	Year	Marks
Q.1	Solve the following equations by crouts method 10x+y+z=12 , 2x+10y+z=13 , 2x+2y+10z=14	RGPV DEC 2013	7
Q.2	Solve the following equations by the application of crouts triangularization(LU) method : 2x-3y+10z=3, -x+4y+2z=20 , 5x+2y+z=-12	RGPV DEC 2010	7

Jacobi's method

2. ITERATIVE METHODS OR INDIRECT METHODS :-

We start with an approximation to the true solution and by applying the method repeatedly we get better and better approximation till accurated solution is achieved.

There are two iterative methods for solving simultaneous equation.

a) Jacobi's method (method of successive correction)

b) Gauss-Seidel method (method of successive correction)

2.a) Jacobi's Method

Note - Condition for using the iterative methods is that the coeffients in the leading diagonal are large compared to the other. If these are not so, then on interchanging the equation we can make the leading diagonal dominant diagonal.

EXAMPLE - Solve by Jacobi's method

$$2x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 25$$

(R.G.P.V, Bhopal, June 2009)

Solution - Coeffiecent in the leading diagonal are large compared to the other.

The above equations can be written as.

$$\left.\begin{aligned} x &= \frac{17}{20} - \frac{y}{20} + \frac{2z}{20} \\ y &= \frac{-18}{20} - \frac{3x}{20} + \frac{z}{20} \\ z &= \frac{25}{20} - \frac{2x}{20} + \frac{3y}{20} \end{aligned}\right] \quad \text{----- (1)}$$

Iteration 1. On substituting $x = y = z = 0$ on the right side of (1), we obtain

Iteration 2. $x = \frac{17}{20}$, $y = \frac{-18}{20}$, $z = \frac{25}{20}$

Iteration 3. Again substituting the value of x, y and z on R.H.S of (1), we get.

$$x = \frac{17}{20} + \frac{18}{400} + \frac{50}{400} = \frac{408}{400} = 1.02$$

$$y = \frac{-18}{20} - \frac{51}{400} + \frac{25}{400} = \frac{-386}{400} = -0.965$$

$$z = \frac{25}{20} - \frac{34}{400} - \frac{54}{400} = \frac{412}{400} = 1.03$$

Iteration 4. Repeating the process for $x = 1.02$, $y = -0.965$ and $z = 1.03$, we have

$$x = 0.85 + \frac{0.965}{20} + 2\frac{(1.03)}{20} = 1.00125$$

$$y = -0.9 - 3\frac{(1.02)}{20} + \frac{1.03}{20} = -1.0015$$

$$z = 1.25 - 2\frac{(1.02)}{20} + 3\frac{(-0.965)}{20} = 1.00325$$

This can be written in the following table.

Iteration	1	2	3	4	5
$x = \frac{17}{20} - \frac{y}{20} + \frac{2z}{20}$	0	0.85	1.02	1.00125	1.0004
$y = \frac{-18}{20} - \frac{3x}{20} + \frac{z}{20}$	0	−0.9	−0.965	−1.0015	−1.000025
$z = \frac{25}{20} - \frac{2x}{20} + \frac{3y}{20}$	0	1.25	1.03	1.00325	0.99964

After 5th iteration $x = 1.0004$ $y = -1.000025$, $z = 0.99964$

The actual values $x = 1$, $y = -1$ $z = 1$

S.NO	RGPV QUESTIONS	Year	Marks
Q.1	Solve the following equations by crouts method 10x+y+z=12 , 2x+10y+z=13 , 2x+2y+10z=14	RGPV DEC 2013	7
Q.2	Solve the following equations by the application of crouts triangularization(LU) method : 2x-3y+10z=3, -x+4y+2z=20 , 5x+2y+z=-12	RGPV DEC 2010	7

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Gauss-seidel iterative method

2.b) GAUSS – SEIDEL METHOD –

Gauss – Seidel method is a modification of Jacobi's method.

Note :- 1) The convergence of Gauss- Seidal method is twice as fast as in Jacobi's method.

2) If the absolute value of largest coefficent is greater than the sum of the absolute value of all the remaining coefficent than the method

EXAMPLE : Apply Gauss – Seidel method to solve.

$$5x + 2y + z = 12$$
$$x + 4y + 2z = 15$$
$$x + 2y + 5z = 20$$

Correct upto two decimal places, taking

$x_0 = y_0 = z_0 = 0$ (R.G.P.V, Bhopal III Sem Feb 2010)

SOLUTION : The given equations can be written as

$$x = \frac{12}{5} - \frac{2}{5}y - \frac{z}{5} \quad \text{---- (1)}$$
$$y = \frac{15}{4} - \frac{x}{4} - \frac{z}{2} \quad \text{------ (2)}$$
$$z = 4 - \frac{x}{5} - \frac{2y}{5} \quad \text{----- (3)}$$

Putting $y = z = 0$ in (1), we have $x = 2.4$

Putting $= 2.4$ $z = 0$ in (2), we have

$$y = \frac{15}{4} - \frac{2.4}{4} - 0 = 3.15$$

Putting $x = 2.4$, $y = 3.15$ in (3), we have

$$z = 4 - \frac{2.4}{5} - \frac{2 \times 3.15}{5} = 2.26$$

Again starting from (1) and Putting $y = 3.15$, $z = 2.26$ we ge

$$x = \frac{12}{5} - \frac{2}{5} \times 3.15 - \frac{2.26}{5} = 0.688$$

Similarly the process is carried on the roots so obtained are given in the following table :

Iterations	1	2	3	4	5
$x = \frac{12}{5} - \frac{2}{5}y - \frac{z}{5}$	2.4	0.688	0.84416	0.962612	0.99426864
$y = \frac{15}{4} - \frac{x}{4} - \frac{z}{2}$	3.15	2.448	2.09736	2.013237	2.00034144
$x = 4 - \frac{x}{5} - \frac{2y}{5}$	2.26	2.8832	2.99222	3.0021828	3.001009696

Hence, exact roots are $x = 1$, $y = 2$, $z = 3$

S.NO	RGPV QUESTIONS	Year	Marks
Q.1	Solve the following equations by gauss seidal method 20x+y-2z=17 , 3x+20y-z=-18 , 2x-3y+20z=25	RGPV DEC 2013	7
Q.2	Solve the following equations by gauss seidal method : 8x-y+z=18, 2x+5y-2z=3 , x+2y-3z=-6	RGPV DEC 2014	7
Q. 3	Solve the following equations by using gauss seidal method : 27x+6y-z=85, 6x+15y+2z=72 , x+y+54z=110	RGPV JUNE 2011	7

Unit-02/Lecture-10
Error and approximations

Absolute error, relative error and percentage errors.

Absolute error is the numerical difference between the true value and approximate value of a quantity.
If X is true value and X′ is approximate value then X —X′ is called absolute error.

$$e_a = |X - X'|$$

Relative error $e_r = \left|\frac{X - X'}{X}\right|$

Percentage error $e_p = 100 \times e_r = 100 \times \left|\frac{X - X'}{X}\right|$

Truncation and round off error in numerical calculations.

Truncation errors : They are caused by using approximate results or on replacing
an infinite process by a finite one.

e.g. If $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots\ldots \infty = X$ (say) is truncated to $1 - \frac{x^2}{2!} + \frac{x^4}{4!} = X'$ say, then truncation error = $X - X'$.

Round off errors : They arise when a calculated figure is rounded off to a fixed
number of digits to represent the exact figure, the difference between such rounded figure and exact figure is called round off error.
e.g. Round off 35.46735 correct to four significant digits.
Number rounded off to four significant digits = 35.47.

Types of errors arise in numerical calculations.

In numerical calculations we come across following types of errors
(a) Inherent errors
(b) Rounding errors
(c) Truncation errors
(d) Absolute errors
(e) Relative errors
(J) Percentage errors.

Q. 1.4. Explain rounding off errors and explain how to reduce it.

Sol. Rounding off errors-These errors arise from The process of rounding off the numbers during the computation. These errors can be reduced:
(a) By changing the calculation procedure so as to avoid subraction of nearly equal number or division by a small number.
(b) By retairing at least one more significant digit at each step and rounding off at last step.

Q. 1.5. Define absolute, relative and percentage errors by giving suitable example.

Sol. Absolute error:

$$e_a = |X - X'|$$

X – True value; X′ – approximate value

Relative error :

$$e_r = \left|\frac{X - X'}{X}\right|$$

X – true value; X′ – approximate value

Percentage error :

$$e_p = \left|\frac{X - X'}{X}\right| \times 100 = e_r \times 100$$

e.g. **If true value** $= \frac{10}{3} = X$ app. value $= 3.33 = X'$

Absolute error $e_a = |X - X'| = 0.003333$

Relative error $e_r = \left|\frac{X - X'}{X}\right| = 0.000999$

Percentage error $e_p = 0.000999 \times 100 = 0.0999$

Note : 1. If a no. is correct to n decimal places then error $= \frac{1}{2} \times 10^{-n}$

2. If the first significant figure of no. is K and the no. is correct to n significant figures, then **relative error** $< \frac{1}{K \times 10^{n-1}}$

Q. 1.6. Find the no. of terms of logarithm series such that value of log 1.02 is correct to three decimals.

Sol. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \ldots\ldots\ldots\ldots$

If we retain n terms then $(n+1)^{\text{th}}$ term $= (-1)^{n+1} \frac{x^n}{n}$.

For determination of log (1.02) correct to three decimals at $x = 0.02$.

$\left|\frac{(-1)^{n+1}(0.02)^n}{n}\right| < \frac{1}{2} \times 10^{-3}$ where $n = 3$ satisfies.

Q. 1.7. The error in the measurement of area of circle is not allowed to exceed 0.1 %• How accurately should the diameter be measured?

Sol. Let $A = \frac{\pi d^2}{4}$

A = area

d = diameter

Taking log on both sides

$$\log A = \log \pi + 2 \log d - \log 4$$

Differentiating ,

$$\frac{\delta A}{A} \times 100 = \frac{2}{d}(\delta d \times 100)$$

$$\Rightarrow \quad \frac{\delta d}{d} \times 100 = \frac{0.1}{2} = 0.05.$$

Q. 1.14. Differentiate inherent errors and Truncation errors.

Ans. Inherent Errors : The interent error is that quantity which is already present in the statement of problem before its solution.

These errors arise either due to limitations of Mathematical tables, calculators and computers.

The inherent errors further classified as

(a) Data Errors (b) Conversion Error.

Truncation Error : The truncation error arised by using approximate results or on replacing infinite process by finite one. Suppose we have

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \ldots\ldots\infty = X(\text{say})$$

Let series truncated to

$$X' = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$$

Trunction error $= |X - X'|$

Reference

Book	Author
Higher Engg. Mathematics	B.S.Grewal
Engg. Mathematics - III	Dr. D.C.Agarwal