UNIT - 3

Unit-03/Lecture-01

Difference operators

Let $x_0, x_1, x_2 \ldots x_n$ be the values of independent data which have been collected at equal or unequal interval and $y_0, y_1, y_2 \ldots y_n$ be the values of dependent variables for corresponding values of x_i.

In case x_i are at regular interval h then we handle the data with finite difference operators Δ (called the forward operator), ∇ (called the backward operator), δ (called the central operator), E (called the shift operator) and μ (called the averaging operator).

These operators are defined as follows:

1. $\Delta y_0 = y_1 - y_0, \quad \Delta y_r = y_{r+1} - y_r$, etc.
2. $\nabla y_n = y_n - y_{n-1}, \quad \nabla y_1 = y_1 - y_0$, etc. $\qquad$ (2.1)
3. $E y_0 = y_1, \quad E\, y_r = y_{r+1}$, etc.
4. $\delta(f(x)) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)$ or $\delta f(x) = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) f(x)$ or
5. $\mu y_r = \frac{1}{2}\left(y_{r+\frac{1}{2}} + y_{r-\frac{1}{2}}\right)$ or $\mu f(x) = \frac{1}{2}\left(f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right) f(x)$

$$= \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) f(x) \quad \text{or} \quad \mu = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$$

The second-order operators are as follows:

1. $\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$
$$= (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$
2. $\nabla^2 y_n = \nabla(\nabla y_n) = \nabla(y_n - y_{n-1}) = \nabla y_n - \nabla y_{n-1} = (y_n - y_{n-1}) - (y_{n-1} - y_{n-2})$
$$= y_n - 2y_{n-1} + y_{n-2} \qquad (2.2)$$
3. $E^2 y_0 = E(Ey_0) = Ey_1 = y_2$
4. $\delta^2 y_x = \delta\left(y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}\right) = \delta y_{x+\frac{h}{2}} - \delta y_{x-\frac{h}{2}}$
$$= (y_{x+h} - y_x) - (y_x - y_{x-h}) = y_{x+h} - 2y_x + y_{x-h}$$

2.4 RELATIONSHIP BETWEEN THE OPERATORS

All the operators, Δ, ∇, δ, μ and D can be expressed in terms of E and the relations are taken as standard results. These relations are of great use in the development of theory of finite difference and solving problems.

1. ∇ in terms of E

$$\begin{aligned} \Delta y_n &= y_{n+1} - y_n \\ &= Ey_n - y_n \\ &= (E - 1)\, y_n \end{aligned}$$

$$\therefore \Delta = E - 1 \quad \text{or} \quad E = 1 + \Delta \qquad (2.3)$$

2. ∇ in terms of E

$$\nabla y_{n+1} = y_{n+1} - y_n = y_{n+1} - E^{-1} y_{n+1} = (1 - E^{-1})\, y_{n+1}$$

$$\therefore \nabla = 1 - E^{-1} \quad \text{or} \quad E = (1-\nabla)^{-1} \qquad (2.4)$$

3. δ in terms of E

By definition $\delta y_r = y_{r+h/2} - y_{r-h/2}$

$$= E^{1/2} y_r - E^{-1/2} y_r = (E^{1/2} - E^{-1/2})\, y_r$$

$$\therefore \delta = E^{1/2} - E^{-1/2}. \qquad (2.5)$$

4. μ in terms of E

$$\mu y_r = \frac{1}{2}\left[y_{r+\frac{h}{2}} + y_{r-\frac{h}{2}} \right] = \frac{1}{2}\left[E^{\frac{1}{2}} y_r + E^{-\frac{1}{2}} y_r \right] = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right] y_r$$

$$\therefore \mu = \frac{1}{2}\left[E^{\frac{1}{2}} + E^{-\frac{1}{2}} \right] \qquad (2.6)$$

5. D in terms of E

$$Ef(x) = f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \ldots \text{ (by Taylor's series)}$$

$$= \left[1 + hD + \frac{h^2}{2!} D^2 + \ldots \right] f(x) \quad (f'(x) = Df(x))$$

$$\therefore E = 1 + hD + \frac{h^2}{2!} D^2 + \ldots$$

$$= e^{hD}$$

$$\text{or} \quad D = \frac{1}{h}\log E = \frac{1}{h}\log(1-\nabla)^{-1} = \frac{1}{h}\log(1+\Delta) \qquad (2.7)$$

2.4.1 Some Interrelationships in Operators

1. $E = 1 + \Delta$ also $E = (1 - \nabla)^{-1}$

   $\therefore\ 1 + \Delta = (1 - \nabla)^{-1}$ or $(1 + \Delta)(1 - \nabla) = 1$

   or $1 - \Delta\nabla + \Delta - \nabla = 1$

   or $\Delta\nabla = \Delta - \nabla$

Also $\Delta\nabla = (E - 1)(1 - E^{-1}) = \dfrac{(E-1)^2}{E}$

$$= E - 2 + E^1$$

$$= \left(E^{1/2} - E^{-1/2}\right)^2 = \delta^2$$

$$\Delta\nabla = \Delta - \nabla = \delta^2$$

2. $D = \frac{1}{h}\log E = \frac{1}{h}\log(1+\Delta) = \frac{1}{h}\left[\Delta - \frac{\Delta^2}{2} + \frac{\Delta^3}{3} - \dots\right]$

3. $D = \frac{1}{h}\log E = \frac{1}{h}\log(1-\nabla)^{-1} = \frac{1}{h}\left[\nabla + \frac{\nabla^2}{2} + \frac{\nabla^3}{3} - \dots\right]$

4. $\delta E^{1/2} = \left(E^{1/2} - E^{-1/2}\right)E^{1/2} = E - 1 = \Delta$

5. $\mu\delta = \frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) = \frac{1}{2}\left[E - E^{-1}\right]$

$= \frac{1}{2}((1+\Delta) - (1-\nabla)) = \frac{1}{2}(\Delta + \nabla)$

6. $\mu - \frac{\delta}{2} = \frac{E^{1/2} + E^{-1/2}}{2} - \frac{E^{1/2} - E^{-1/2}}{2} = E^{-1/2}$

7. $\nabla = 1 - E^{-1} = 1 - e^{-hD}$

8. $\Delta = \mu\delta + \frac{1}{2}\delta^2$

RHS is $\frac{1}{2}\left(E^{1/2} + E^{-1/2}\right)\left(E^{1/2} - E^{-1/2}\right) + \frac{1}{2}\left(E^{1/2} - E^{-1/2}\right)^2$

$= \frac{1}{2}\left(E - E^{-1}\right) + \frac{1}{2}\left(E + E^{-1} - 2\right)$

$= E - 1$

$= \Delta$ (LHS)

9. $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

RHS $= \frac{\Delta^2 - \nabla^2}{\nabla\Delta} = \frac{(\Delta+\nabla)(\Delta-\nabla)}{\Delta\nabla}$

By (1) We have $\Delta - \nabla = \Delta\nabla$

$\therefore \Delta + \nabla =$ LHS

10. $\delta = \Delta(1+\Delta)^{-1/2}$

RHS $(E-1)E^{-1/2} = E^{1/2} - E^{-1/2} = \delta$

Example 2.4 Prove that $\Delta \log f(x) = \log\left(1 + \frac{\Delta f(x)}{f(x)}\right)$.

Solution

LHS $= \Delta \log f(x) = \log f(x+h) - \log f(x)$

$$= \log\left(\frac{f(x+h)}{f(x)}\right) = \log\left\{\frac{f(x) + f(x+h) - f(x)}{f(x)}\right\}$$

$$= \log\left(\frac{f(x) + \Delta f(x)}{f(x)}\right) = \log\left(1 + \frac{\Delta f(x)}{f(x)}\right) \text{ Proved.}$$

Example 2.5 Evaluate $\Delta^3\left(\frac{1}{x}\right)$.

Solution

$$\Delta\left(\frac{1}{x}\right)=\frac{1}{x+h}-\frac{1}{x}=\frac{-h}{(x+h)x}$$

$$\Delta^2\left(\frac{1}{x}\right)=\Delta\left(\frac{-h}{(x+h)x}\right)=\frac{(-1)^2}{(x+2h)(x+h)}\frac{1.2h^2}{x}$$

$$\Delta^3\left(\frac{1}{x}\right)=\frac{(-1)^3 3!h^3}{x(x+h)(x+2h)}$$

$$\therefore \Delta^n\left(\frac{1}{x}\right)=\frac{(-1)^n n!h^n}{x(x+h)(x+2h)\ldots(x+nh)}$$

In case $h=1$ $\Delta^n\left(\frac{1}{x}\right)=\frac{(-1)^n n!}{x(x+1)(x+2)\ldots(x+n)}$

.

Example 2.10 Find $\Delta(x^2 e^{3x})$.

Solution

By Eq. (2.8)

$$\Delta\left(x^2e^{3x}\right)=(x+h)^2\left[e^{3(x+h)}-e^{3x}\right]+e^{3x}\left[\left(x+h\right)^2-x^2\right]$$
$$=(x+h)^2\ \left(e^{3h}-1\right)\ e^{3x}+e^{3x}\left(h^2+2xh\right).$$

Example 2.11 Find $\Delta\left(\frac{e^{ax}}{\log x}\right)$.

Solution

By Eq. (2.9)

$$\Delta\left(\frac{e^{ax}}{\log x}\right)=\frac{\left(e^{a(x+h)}-e^{ax}\right)\ \log x-\left\{\log(x+h)-\log x\right\}\times e^{ax}}{\log(x+h)\log x}$$

$$=\frac{\left(e^{ah}-1\right)\ e^{ax}\ \log x-\log\left(1+\frac{h}{x}\right)e^{ax}}{\log(x+h)\log x}$$

Example 2.13 Evaluate (a) $\Delta\left(\frac{3x+10}{x^2+3x+1}\right)$, and (b) $\frac{\Delta}{E}\left(\frac{3x+10}{x^2+3x+1}\right)$.

Solution

(a) Calculate by Eq. (2.9)

(b) $\frac{\Delta}{E}=\frac{E-1}{E}=\left(1-E^{-1}\right)\left(\frac{3x+10}{x^2+3x+1}\right)$

$$=\frac{3x+10}{x^2+3x+1}-\frac{3(x-1)+10}{(x-1)^2+3(x-1)+1}=\frac{3x+10}{x^2+3x+1}-\frac{3x+7}{x^2+x-1}$$

Example 2.14 Find $\frac{\Delta^2}{E}\left(e^{2x}.x\right), \quad h = 1.$

Solution

$$\therefore \frac{(E-1)^2}{E}\left(e^{2x}.x\right) = \left(E - 2 + E^{-1}\right)\left(e^{2x}.x\right)$$

$$= e^{2(x+1)}(x+1) - 2e^{2x}.x + e^{2(x-1)}(x-1)$$

$$= e^{2x}\left(e^2(x+1) - 2x + e^{-2}(x-1)\right)$$

Example 2.15 Find $\frac{\Delta^2}{E^2}(\cos(x+a))$.

Solution

$$\left(1 - 2E^1 + E^{-2}\right)\cos(x+a)$$

$$= \cos(x+a) - 2\cos(x-h+a) + \cos(x-2h+a)$$

FACTORIAL POLYNOMIAL AND ITS USES IN FINITE DIFFERENCE

The expression $x(x-h)(x-2h)\ldots(x-(n-1)h)$ is called a factorial polynomial of degree n and is denoted by $x^{(n)}$.

The expression $\frac{1}{x(x-h)\ldots(x-(n-1)h)}$ is called reciprocal factorial polynomial and is denoted by $x^{(-n)}$.

An ordinary polynomial may be changed to factorial polynomial by synthetic division and the resulting polynomial may be used for finding any difference of the polynomial easily.

Synthetic Division and Factorial Polynomial from Ordinary Polynomial

Let $a_0x^4 + a_1x^3 + a_2x^2 + a_4$ be the ordinary polynomial and h be interval.

1. Write the coefficients in descending powers of x. Take zero for missing term.

   $a_0 \quad a_1 \quad a_2 \quad 0 \quad a_4$

2. Multiply a_0 by zero and place it below a_1, add the result and multiply the sum by zero and place it below a_2 and complete the line.

$0 \lrcorner$	a_0	a_1	a_2	0	a_4
		0	0	0	0
	–	–	–	–	–
	a_0	a_1	a_2	0	$[a_4]$

   a_4 is the last coefficient of factorial polynomial.

3. Repeat the process by h

$h \lrcorner$	a_0	a_1	a_2	0	$[a_4]$
		a_0h	b_1h	c_1h_2	
	–	–	–	–	–
	a_0	b_1	c_1	$[\boldsymbol{d}_1]$	

   $\boldsymbol{d}_1$ is the coefficient of $x^{(1)}$

4. Repeat the process with $2h$ and $3h$, respectively

$2h \lrcorner$	a_0	b_1	c_1	$[\boldsymbol{d}_1]$
		a_02h	d_2h	
	a_0	(d_2)	$[\boldsymbol{d}_3]$	
$3h \lrcorner$	a_0	d_2	$[\boldsymbol{d}_3]$	
		$3ha_0$		
	–	–		
	$[\boldsymbol{a}_0]$	$[\boldsymbol{e}_1]$		

   The factorial polynomial is

   $a_0x^{(4)} + e_1x^{(3)} + d_3x^{(2)} + d_1x^{(1)} + a_4.$

Unit-03/Lecture-02

Example 2.16 Find the factorial polynomial of the ordinary polynomial $4x^4 - 3x^3 - x + 5$ when $h = 1$.

Solution

0..1	4	−3	0	−1	5	
			0	0	0	0
	1⌋	4	−3	0	−1	**[5]**
		–	–	–	–	
			4	1	1	
		----	----	----		
	2⌋	4	1	1	**[0]**	
	–	–	–	–		
			8	18		
			----	----		
		4	9	**[19]**		
		–	–	–		
	3⌋		12			
		[4]	**[21]**			
	–	–				

The factorial polynomial of the given polynomial is

$$4x^{(4)} + 21\,x^{(3)} + 19\,x^{(2)} + 0\,x^{(1)} + 5$$

The meaning is that the given ordinary polynomial may be written as

$$4x(x-1)\,(x-2)\,(x-3) + 21\,x\,(x-1)\,(x-2) + 19\,x\,(x-1) + 5$$

Example 2.17 Find the factorial polynomial of the polynomial for $h = 2$

$$2x^5 - x^4 + 2x^2 + 3x + 1.$$

Solution

0	2	−1	0	2	3	1
		0	0	0	0	0
	–	–	–	–	–	–
$h = 2$	2	−1	0	2	3	**[1]**
		4	6	12	28	
	–	–	–	–	–	
$2h = 4$	2	3	6	14	**[31]**	
		8	44	200		
	–	–	–	–		
$3h = 6$	2	11	50	**[214]**		
		12	138			
	–	–	–			
$4h = 8$	2	23	**[188]**			
		16				
	–	–				
	[2]	**[39]**				

The factorial polynomial is

$$2x^{(5)} + 39x^{(4)} + 188x^{(3)} + 214x^{(2)} + 31x^{(1)} + 1$$

S.NO	RGPV QUESTIONS	Year	Marks
Q.1	Prove that $e^x = \left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{Ee^x}{\Delta^2 e^x}$	RGPV,JUNE 2014	2
Q. 2	Prove that (I) $e^x = \left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{Ee^x}{\Delta^2 e^x}$ (II) $e^{hD} = 1 + \Delta$	RGPV,JUNE 2011	7
Q . 3	Prove with the usual notations that (I)($E^{\frac{1}{2}} + E^{\frac{-1}{2}})(1+\Delta)^{\frac{1}{2}} = 2 + \Delta$ (II) Express $y = 3x^3 + x^2 + x + 1$ in factorial function and hence show that $\Delta^3 y = 18$	RGPV,DEC 2010	7

Unit-03/Lecture-03

Interpolation(Newton forward)

Consider the points x_j, $x_j + h$, $x_j + 2h$, . . ., and recall that

$$Ef_j = f_{j+1} = f(x_j + h), \quad E^{\theta} f_j = f_{j+\theta} = f(x_j + \theta h),$$

where θ is any real number. Formally, one has (since $\Delta = E - 1$)

$$\begin{aligned} f(x_j + \theta h) &= E^{\theta} f_j \\ &= (1+\Delta)^{\theta} f_j \\ &= \left[1 + \theta\Delta + \frac{\theta(\theta-1)}{2!}\Delta^2 + \frac{\theta(\theta-1)(\theta-2)}{3!}\Delta^3 + \cdots\right] f_j, \end{aligned}$$

which is **Newton's forward difference formula**. The linear and quadratic (forward) interpolation formulae correspond to first and second order truncation, respectively. If we truncate at n-th order, we obtain

$$f(x_j + \theta h) \approx \left[1 + \theta\Delta + \frac{\theta(\theta-1)}{2!}\Delta^2 + \cdots + \frac{\theta(\theta-1)\cdots(\theta-n+1)}{n!}\Delta^n\right] f_j$$

$$\Delta^{n+k} f_j = 0, \quad k = 1, 2, \ldots$$

which is the case if f is a polynomial of degree n.

example :- consider the difference table of $f(x) = \sin x$ for $x = 0°(10°)50°$:

$x°$	$f(x) = \sin x$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
0	0					
		1736				
10	0.1736		−52			
		1684		−52		
20	0.3420		−104		4	
		1580		−48		0
30	0.5000		−152		4	
		1428		−44		
40	0.6428		−196			
		1232				
50	0.7660					

Since the fourth order differences are constant, we conclude that a quartic approximation is appropriate. (The third-order differences are not quite constant within expected round-offs, and we anticipate that a cubic approximation is not quite good enough.) In order to determine sin 5° from the table, we use Newton's forward difference formula (to fourth order); thus, taking $x_j = 0$, we find $\theta = \frac{5-0}{10} = \frac{1}{2}\ (h = 10)$, and

$$\sin 5^\circ \approx \sin 0^\circ + \tfrac{1}{2}(0.1736) + \tfrac{1}{2}\tfrac{1}{2}\left(-\tfrac{1}{2}\right)(-0.0052) + \tfrac{1}{6}\tfrac{1}{2}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)(-0.0052)$$
$$+ \tfrac{1}{24}\tfrac{1}{2}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)\left(-\tfrac{5}{2}\right)(0.0004)$$
$$= 0 + 0.0868 + 0.0006(5) - 0.0003(3) - 0.0000(2)$$
$$= 0.0871 \text{ (compare with the actual value 0.0872 to } 4D)$$

MISSING TERM METHOD:-

Example 2.20 Find the missing term.

x	1	2	3	4	5
y	−1	3	–	53	111

Solution

First Method

The number of data in y is 4. So y will fit a third degree polynomial in x.

$$\therefore \Delta^4 y_0 = 0 \text{ or } (E-1)^4 y_0 = 0$$

or $\left(E^4 - 4E^3 + 6E^2 - 4E + 1\right) y_0 = 0$

or $y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$ (2.10)

From the table, we see that $y_0 = -1$, $y_1 = 3$, $y_2 = ?$, $y_3 = 53$, $y_4 = 111$.

Putting these values in Eq. (2.10), we get

$$111 - 4 \times 53 + 6y_2 - 4 \times 3 - 1 = 0$$
or $y_2 = 19$

This result is correct because the data satisfy the cubic $f(x) = x^3 - 3x + 1$ and $f(3) = 19$.

This method gives correct result and labor is much less.

Second Method

The other way of handling the problem is to make the difference table

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	−1				
		4			
2	3		$y_2 - 7$		
		$y_2 - 3$		$63 - 3y_2$	
3	y_2		$56 - 2y_2$		$6y_2 - 114$
		$53 - y_2$		$3y_2 - 51$	
4	53		$5 + y_2$		
		58			
5	111				
6					

Since we have only four values of y, the data $\Delta^4 y_0 = 0$,

i.e., $6y_2 - 114 = 0$ or $y_2 = 19$.

We conclude that first method is better and has less work.

Unit-03/Lecture-03

Example 2.21 Find the missing data.

x	0	5	10	15	20	25
y	6	10	–	17	–	31
	y_0	y_1	y_2	y_3	y_4	y_5

Solution

The number of data in y is 4, so the data will fit a cubic polynomial, then

1. $\Delta^4 y_0 = 0$ and 2. $\Delta^4 y_1 = 0$ (for two missing data take two equations).
   First equation gives $(E-1)^4 y_0 = 0$
   or $y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$
   or $y_4 - 4 \times 17 + 6y_2 - 4 \times 10 + 6 = 0$, giving $y_4 + 6y_2 = 102$ (i)
   The second equation $\Delta^4 y_1 = 0$ gives
   $(E-1)^4 y_1 = 0$ or $y_5 - 4y_4 + 6y_3 - 4y_2 + y_1 = 0$
   Putting the values of y from the table, we get
   $31 - 4y_4 + 6 \times 17 - 4y_2 + 10 = 0$
   or $4y_2 + 4y_4 = 143$ (ii)
   Now solving (i) and (ii), we get $y_2 = 13.25$ and $y_4 = 22.50$.

S.NO	RGPV QUESTIONS	Year	Marks
Q.1	Derive Newton's forward interpolation formula.	JUNE 2014	2
Q. 2	Define interpolation and write the Newton's Forward and Backward interpolation formula.	DEC 2014	2
Q. 3	Find f(9) from the following table: X: 5, 7, 11, 13, 17 F(x): 150, 392, 1452, 2366, 5202	JUNE 2011	7
Q. 4	Find the number of mem getting wages between Rs. 10 and Rs. 15 from the following data: Wages in (Rs.): 0-10, 10-20, 20-30, 30-40 Frequency: 9, 30, 35, 42	RGPV DEC 2013, JUNE 2010	7

Unit-03/Lecture-04

Backward formula

1. **Newton's backward difference formula**

Formally, one has (since $\nabla = 1 - E^{-1}$)

which is **Newton's backward difference formula**. The linear and quadratic (backward) interpolation formulae correspond to truncation at first and second order, respectively. The approximation based on the $f_{j-n}, f_{j-n+1}, \dots, f_{j-1}, f_j$ is

$$f(x_j + \theta h) \approx \left[1 + \theta\nabla + \frac{\theta(\theta+1)}{2!}\nabla^2 + \dots + \frac{\theta(\theta+1)\cdots(\theta+n-1)}{n!}\nabla^n\right] f_j$$

Newton-Gregory Backward Difference Interpolation polynomial:

If the data size is big then the divided difference table will be too long. Suppose the desired intermediate value $(\tilde{x})$ at which one needs to estimate the function $(i.e. f(\tilde{x}))$ falls towards the end or say in the second half of the data set then it may be better to start the estimation process from the last data set point. For this we need to use backward-differences and backward difference table. (11.1)

Let us first define backward differences and generate backward difference table, say for the data set $(x_i, f_i), i = 0, 1, 2, 3, 4.$

First order backward difference ∇f_i is defined as:

$$\nabla f_i = f_i - f_{i-1}$$

Second order backward difference $\nabla^2 f_i$ is defined as:

$$\nabla^2 f_i = \nabla f_i - \nabla f_{i-1} \tag{11.2}$$

In general, the k^{th} order backward difference is defined as

$$\nabla^k f_i = \nabla^{k-1} f_i - \nabla^{k-1} f_{i-1} \tag{11.3}$$

In this case the reference point is x_n and therefore we can derive the Newton-Gregory backward difference interpolation polynomial as:

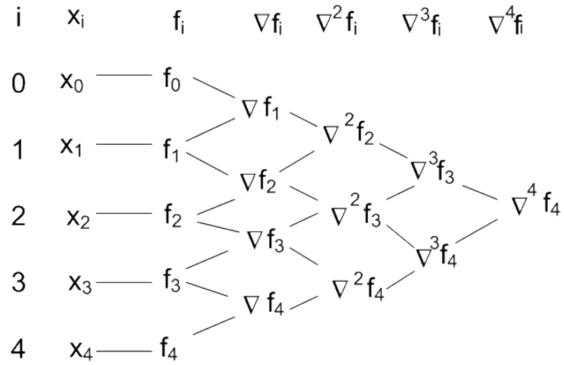
$$P_n(S) = f_n + s\nabla f_n + \frac{s(s+1)}{2!}\nabla^2 f_n + \dots\dots + \frac{s(s+1)\dots(s+n-1)}{n!}\nabla^n f_n \tag{12}$$

Where $s = \frac{x - x_n}{h}$

For constructing $P_n(s)$ as given in $Eqn.(12)$ it will be easier if we first generate backward-difference table.

The backward difference table for the data $(x_i, f_i), \quad i = 0, 1, 2, 3, 4$ is given below:

Unit-03/Lecture-04



Example: Given the following data estimate $f(4.12)$ using Newton-Gregory backward difference interpolation polynomial:

i	0	1	2	3	4	5
x_i	0	1	2	3	4	5
f_i	1	2	4	8	16	32

Solution:
Here

$$x_n = 5, \quad x = 4.12, \qquad h = 1$$

$$\therefore s = \frac{x - x_n}{h} = \frac{4.12 - 5}{1} = -0.88$$

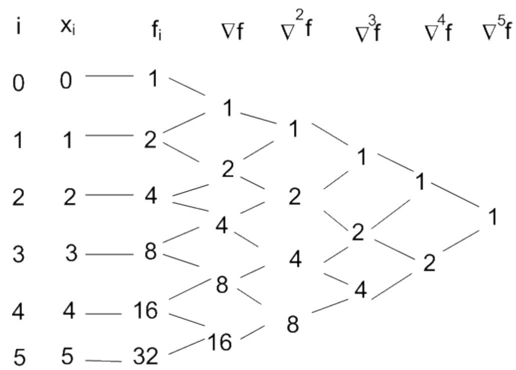
$\therefore$Newton Backward Difference polynomial $P_5(x)$ is given by

$$P_5(s) = f_5 + s\nabla f_5 + \frac{s(s+1)}{2!}\nabla^2 f_5 + \frac{s(s+1)(s+2)}{3!}\nabla^3 f_5 + \frac{s(s+1)(s+2)(s+3)}{4!}\nabla^4 f_5 +$$

$$\frac{s(s+1)(s+2)(s+3)(s+4)}{5!}\nabla^5 f_5$$

Unit-03/Lecture-04

Let us first generate backward difference table:



$$\therefore P_5(-0.88) = 32 + (-0.88)16 + \frac{(-0.88)(-0.88+1)}{2}8 + \frac{(-0.88)(-0.88+1)(-0.88+2)}{6}(4)$$

$$+\frac{(-0.88)(-0.88+1)(-0.88+2)(-0.88+3)}{24} \quad (2)\frac{+(-0.88)(-0.88+1)(-0.88+2)(-0.88+3)(-0.88+4)}{120} \qquad .1$$

$$= 32 - 14.08 - 0.4224 - 0.07885 - 0.0209 - 0.0065$$

$$= 17.92 - 0.4229 - 0.7885 - 0.0209 - 0.0065$$

$$= 17.4976 - 0.07885 - 0.0209 - 0.0065$$

$$= 17.41875 - 0.0209 - 0.0065$$

=17.39135 (13.5)

S.NO	RGPV QUESTIONS	Year	Marks
Q.1	Define interpolation and write the Newton's Forward and Backward interpolation formula.	RGPV DEC 2014	2

Unit-03/Lecture-05

Central interpolation formula

→ STIRLING FORMULA (CENTRAL DIFFERENCE)

This formula is applied for interpolation near the middle value of the table.

$$y_n = y_0 + p\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \frac{p(p^2-1^2)}{3!}\mu\delta^3 y_0 + \frac{p^2(p^2-1^2)}{4!}\delta^4 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\mu\delta^5 y_0 + \ldots$$

y	δ	δ^2	δ^3	δ^4
y_{-2}				
	$\delta y_{-3/2}$			
y_{-1}		$\delta^2 y_{-1}$		
	$\delta y_{-1/2}$		$\delta^3 y_{-1/2}$	
y_0		$\delta^2 y_0$		$\delta^4 y_0$
	$\delta y_{1/2}$		$\delta^3 y_{1/2}$	
y_1		$\delta^2 y_1$		
	$\delta y_{3/2}$			
y_2				

$$\mu\delta y_0 = \frac{1}{2}\left(\delta y_{-\frac{1}{2}} + \delta y_{\frac{1}{2}}\right)$$

$$\mu\delta^3 y_0 = \frac{1}{2}\left(\delta^3 y_{-\frac{1}{2}} + \delta^3 y_{\frac{1}{2}}\right)$$

From the central table, we have

$\frac{1}{2}(\Delta y_0 + \Delta y_{-1}) = \frac{1}{2}(\delta y_{1/2} + \delta y_{-1/2}) = \mu\delta y_0,$

$\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) = \frac{1}{2}(\delta^3 y_{1/2} + \delta^3 y_{-1/2}) = \mu\delta^3 y_0$ etc.

$$\Rightarrow y_p = y_0 + p\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \frac{p(p^2-1^2)}{3!}\mu\delta^3 y_0 + \frac{p^2(p^2-1^2)}{4!}\delta^4 y_0 + \frac{p(p^2-1^2)(p^2-2^2)}{5!}\delta^5\mu y_0 + \ldots$$

Note: This formula involves even differences and mean of odd differences.

EXAMPLE : Use stirling's formula to evaluate $f(1.22)$, given :

x	1.0	1.1	1.2	1.3
$f(x)$	8.403	8.781	9.129	9.451

(R.G.P.V., Bhopal III Sem June 2007)

SOLUTION : Difference table is

x	p	$y = f(x)$	$\delta f(x)$	$\delta^2 f(x)$	$\delta^3 f(x)$
1.0	-2	8.403			
			0.378		
1.1	-1	8.781		-0.030	
			0.348		0.004
1.2	0	9.129		-0.026	
			0.322		
1.3	1	9.451			

$a + ph = 1.22 \qquad a = 1.2, \quad h = 0.1$

$\Rightarrow 1.2 + p(0.1) = 1.22$

$\Rightarrow p = 0.2$

By stirling formula

$$y_p = y_0 + p\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \frac{p(p^2-1^2)}{3!}\mu\delta^3 y_0 + \ldots$$

$$y_{1.22} = 9.129 + 0.2\left(\frac{0.348 + 0.322}{2}\right) + \frac{(0.2)^2}{2}(-0.026)$$

$$= 9.129 + 0.2(0.335) + 0.02(-0.026)$$

$$= 9.129 + 0.0670 - 0.00052 = 9.129 + 0.06648$$

$$= 9.19548$$

Unit-03/Lecture-05

EXAMPLE : Employ Stirling formula to compute $y_{12.2}$ from the following table ($y_x = 1 + \log_{10} \sin x$)

x°	10	11	12	13	14
$10^5 y_x$	23967	28060	31788	35200	38368

(RGPV, Bhopal, III Sem Dec 2005)

SOLUTION : The difference table is.

x	P	$10^5 y_x$	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
10	y_{-2}	23967				
			4093			
11	y_{-1}	28060		−365		
			3728		49	
12	y_0	31788		−316		23
			3412		72	
13	y_1	35200		−244		
			3168			
14	y_2	38368				

$a + ph = 12.2,$ $\qquad a = 12, \; h = 1$

$y_p = y_0 + ph$ $\qquad \Rightarrow 12.2 = 12 + p(1) \quad \Rightarrow p = 0.2$

By stirling formula.

$$y_p = y_0 + p\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \frac{p(p^2-1^2)}{3!}\mu\delta^3 y_0 + \frac{p^2(p^2-1^2)}{4!}\delta^4 y$$

$$10^5 y_{0.2} = 31788 + 0.2\left(\frac{3728+3412}{2}\right) + \frac{(0.2)^2}{2}(-316) + \frac{(0.2)[(0.2)^2-1^2]}{3!}\left(\frac{49+72}{2}\right) + \frac{(0.2)^2[(0.2)^2-1]}{4!}(23)$$

$$= 31788 + (0.2)(3570) + (0.02)(-316) - (0.032)(60.5) - (0.0016)($$

$$= 31788 + 714 - 6.32 - 1.936 - 0.0368 = 32493.7072$$

$$y_{0.2} = 0.324937072$$

S.NO	RGPV QUESTIONS	Year	Marks
Q.1	The following table gives the normal weights of babies during the first 12 months of life <table><tr><td>Age in months:</td><td>0</td><td>2</td><td>5</td><td>8</td><td>10</td><td>12</td></tr><tr><td>Weights in lbs</td><td>$7\frac{1}{2}$</td><td>$10\frac{1}{4}$</td><td>15</td><td>16</td><td>18</td><td>21</td></tr></table>Estimate the weight of the baby at the age of 7 months.	RGPV DEC 2014	7
Q.2	What do you mean by interpolation? The following table gives the amount of a chemical dissolved in water: <table><tr><td>Temperature</td><td>10^0</td><td>15^0</td><td>20^0</td><td>25^0</td><td>30^0</td><td>35^0</td></tr><tr><td>solubility</td><td>19.97</td><td>21.51</td><td>22.57</td><td>23.52</td><td>24.65</td><td>25.89</td></tr></table>Using suitable interpolation estimate the amount dissolved at 22^0 .	RGPV JUNE 2013	7

Unit-03/Lecture-06

Lagrange's formula

The **La-grange's Interpolation Formula** is given as,

$$y = \frac{(x-x_1)(x-x_2)....(x-x_n)}{(x_0-x_1)(x_0-x_2)....(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)....(x-x_n)}{(x_1-x_0)(x_1-x_2)....(x_1-x_n)}y_1 + ... + \frac{(x-x_0)(x-x_1)....(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)....(x_n-x_{n-1})}y_n$$

<u>Example</u> : Compute **f(0.3)** for the data

x	**0**	**1**	**3**	**4**	**7**
f	**1**	**3**	**49**	**129**	**813**

using Lagrange's interpolation formula (Analytic value is **1.831**)

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)} f_0 + \ldots + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)} f_4$$

$$= \frac{(0.3-1)(0.3-3)(0.3-4)(0.3-7)}{(-1)(-3)(-4)(-7)} 1 + \frac{(0.3-0)(0.3-3)(0.3-4)(0.3-7)}{1 \times (-2)(-3)(-6)} 3 + $$

$$\frac{(0.3-0)(0.3-1)(0.3-4)(0.3-7)}{3 \times 2 \times (-1)(-4)} 49 + \frac{(0.3-0)(0.3-1)(0.3-3)(0.3-7)}{4 \times 3 \times 1 (-3)} 129 +$$

$$\frac{(0.3-0)(0.3-1)(0.3-3)(0.3-4)}{7 \times 6 \times 4 \times 3} 813$$

= 1.831

Question 2: Find the value of y at x = 0 given some set of values (-2, 5), (1, 7), (3, 11), (7, 34)?

Solution:

The known values are,

$x = 0$; $x_0 = -2$; $x_1 = 1$; $x_2 = 3$; $x_3 = 7$; $y_0 = 5$; $y_1 = 7$; $y_2 = 11$; $y_3 = 34$

Using the interpolation formula,

$$y = \frac{(0-1)(0-3)(0-7)}{(-2-1)(-2-3)(-2-7)} * 5 + \frac{(0+2)(0-3)(0-7)}{(1+2)(1-3)(1-7)} * 7 + \frac{(0+2)(0-1)(0-7)}{(3+2)(3-1)(3-7)} * 11 + \frac{(0+2)(0-1)(0-3)}{(7+2)(7-1)(7-3)} * 34$$

$$y = \frac{(-1)(-3)(-7)}{(-3)(-5)(-9)} * 5 + \frac{(2)(-3)(-7)}{(3)(-2)(-6)} * 7 + \frac{(2)(-1)(-7)}{(5)(2)(-4)} * 11 + \frac{(2)(1)(3)}{(9)(6)(4)} * 34$$

$$y = \frac{21}{135} * 5 + \frac{42}{36} * 7 + \frac{-14}{40} * 11 + \frac{6}{216} * 34$$

$$y = \frac{21}{27} + \frac{49}{6} + \frac{-77}{20} + \frac{51}{54}$$

$$y = \frac{1087}{180}$$

3. Find **f(2)** for the data **f(0) = 1**, **f(1) = 3** and **f(3) = 55**.

x	**0**	**1**	**3**
f	**1**	**3**	**55**

Solution :

$$f(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f_0 + \ldots + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f_2$$

$$f(2) = \frac{(2 - 1)(2 - 3)}{(0 - 1)(0 - 3)} 1 + \frac{(2 - 0)(2 - 3)}{(1 - 0)(1 - 3)} 3 + \frac{(2 - 0)(2 - 1)}{(3 - 0)(3 - 1)} 55$$

$$f(2) = 21$$

4. Find **f(3)** for

x	**0**	**1**	**2**	**4**	**5**	**6**
f	**1**	**14**	**15**	**5**	**6**	**19**

$$f(3) = \frac{(3 - 1)(3 - 2)(3 - 4)(3 - 5)(3 - 6)}{(0 - 1)(0 - 2)(0 - 4)(0 - 5)(0 - 6)} 1 + \frac{(3 - 0)(3 - 2)(3 - 4)(3 - 5)(3 - 6)}{(1 - 0)(1 - 2)(1 - 4)(1 - 5)(1 - 6)} 14 +$$

$$\frac{(3 - 0)(3 - 1)(3 - 4)(3 - 5)(3 - 6)}{(2 - 0)(2 - 1)(2 - 4)(2 - 5)(2 - 6)} 15 + \frac{(3 - 0)(3 - 1)(3 - 2)(3 - 5)(3 - 6)}{(4 - 0)(4 - 1)(4 - 2)(4 - 5)(4 - 6)} 5 +$$

$$\frac{(3 - 0)(3 - 1)(3 - 2)(3 - 4)(3 - 6)}{(5 - 0)(5 - 1)(5 - 2)(5 - 4)(5 - 6)} 6 + \frac{(3 - 0)(3 - 1)(3 - 2)(3 - 4)(3 - 5)}{(6 - 0)(6 - 1)(6 - 2)(6 - 4)(6 - 5)} 19$$

$$f(3) = 10$$

5. Find **f(0.25)** for

x	0.1	0.2	0.3	0.4	0.5
f	9.9833	4.9667	3.2836	2.4339	1.9177

By Lagrange's formula :

f(0.25) =

$$\frac{(.25-.2)(.25-.3)(.25-.4)(.25-.5)}{(.1-.2)(.1-.3)(.1-.4)(.1-.5)}\,9.9833+\frac{(.25-.1)(.25-.3)(.25-.4)(.25-.5)}{(.2-.1)(.2-.3)(.2-.4)(.2-.5)}\,4.9667+$$

$$\frac{(.25-.1)(.25-.2)(.25-.4)(.25-.5)}{(.3-.1)(.3-.2)(.3-.4)(.3-.5)}\,3.2836+\frac{(.25-.1)(.25-.2)(.25-.3)(.25-.5)}{(.4-.1)(.4-.2)(.4-.3)(.4-.5)}\,2.4339+$$

$$\frac{(.25-.1)(.25-.2)(.25-.3)(.25-.4)}{(.5-.1)(.5-.2)(.5-.3)(.5-.4)}\,1.9177$$

f(0.25) = 3.912

S.NO	RGPV QUESTIONS	Year	Marks
Q.1	Find the cubic polynomial ehich takes the following values: X: 0, 1, 2, 3 F(x): 1, 2, 1, 10 Estimate the weight of the baby at the age of 7 months.	RGPV DEC 2014	2
Q.2	Apply Lagrange's formula to find f(15), if : X: 10, 12, 14, 16, 18, 20 F(x): 2420, 1942, 1497, 1109, 790, 540	RGPV DEC 2010	7

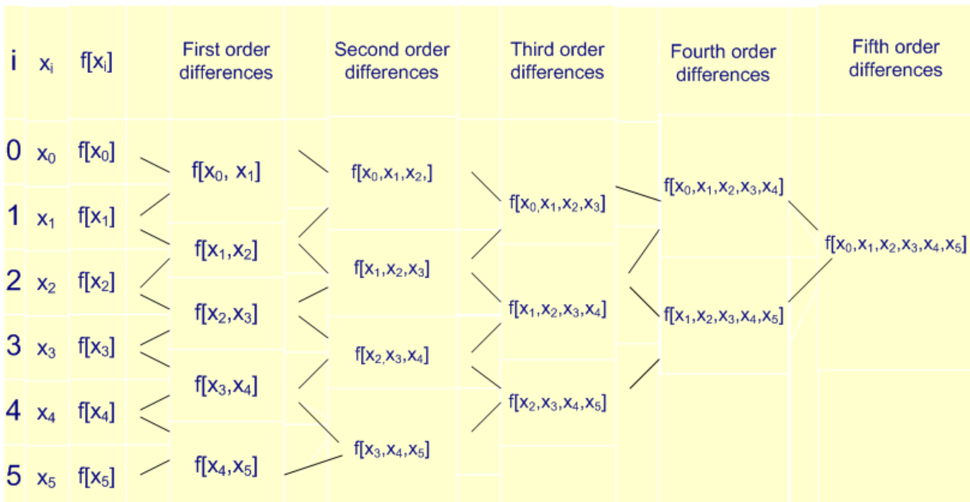
Unit-03/Lecture-07

Divided difference formula

Newton Divided Difference Table:

It may also be noted for calculating the higher order divided differences we have used lower order divided differences. In fact starting from the given zeroth order differences $f[x_i]$; $i = 0,1,...,n,$ one can systematically arrive at any of higher order divided differences. For clarity the entire calculation may be depicted in the form of a table called

Newton Divided Difference Table.



Again suppose that we are given the data set $(x_i, f_i),$ $i = 0......5$ and that we are interested in finding the 5^{th} order Newton Divided Difference interpolynomial. Let us first construct the Newton Divided Difference Table. Wherein one can clearly see how the lower order differences are used in calculating the higher order Divided Differences:

Example: Construct the Newton Divided Difference Table for generating Newton interpolation polynomial with the following data set:

i	0	1	2	3	4
x_i	0	1	2	3	4
$y_i = f(x_i)$	0	1	8	27	64

Solution:

Here $n=5$. One can fit a fourth order Newton Divided Difference interpolation polynomial to the given data. Let us generate Newton Divided Difference Table; as requested.

Unit-03/Lecture-07

i	x_i	$f[x_i]$	1st order differences	2nd order differences	3rd order differences	4th order differences
0	0	0				
			$\frac{1-0}{1-0} = 1$			
1	1	1		$\frac{7-1}{2-0} = 3$		
			$\frac{8-1}{2-1} = 7$		$\frac{6-3}{3-0} = 1$	
2	2	8		$\frac{19-7}{3-1} = 6$		$\frac{1-1}{4-0} = 0$
			$\frac{27-8}{3-2} = 19$		$\frac{9-6}{4-1} = 1$	
3	3	27		$\frac{37-19}{4-2} = 9$		
			$\frac{64-27}{4-3} = 37$			
4	4	64				

Note: One may note that the given data corresponds to the cubic polynomial x^3. To fit such a data 3^{rd} order polynomial is adequate. From the Newton Divided Difference table we notice that the fourth order difference is zero. Further the divided differences in the table can be directly used for constructing the Newton Divided Difference interpolation polynomial that would fit the data.

Exercise: Using Newton divided difference interpolation polynomial , construct polynomials of degree two and three for the following data:

(1) *f(8.1) = 16.94410, f(8.3)=17.56492 , f(8.6) = 18.50515, f(8.7) = 18.82091.*

Also approximate *f(8.4).*

(2) *f(0.6) = -0.17694460 , f(0.7) = 0.01375227 , f(0.8) = 0.22363362 , f(1.0) = 0.65809197.*

Also approximate *f(0.9).*

<table>
<tr><th>S.NO</th><th>RGPV QUESTIONS</th><th>Year</th><th>Marks</th></tr>
<tr><td>Q. 1</td><td>Apply Newton's divided difference formula to find the value of f(9) from the following table:
<table><tr><td>X</td><td>5</td><td>7</td><td>11</td><td>13</td><td>17</td></tr><tr><td>F(x)</td><td>150</td><td>392</td><td>1452</td><td>2368</td><td>5202</td></tr></table></td><td>RGPV JUNE 2014, DEC 2013, JUNE 2010</td><td>7</td></tr>
<tr><td>Q. 2</td><td>Using Newton's divided difference formula to find the value of f(9) from the following table:
<table><tr><td>X</td><td>3</td><td>5</td><td>11</td><td>27</td><td>34</td></tr><tr><td>F(x)</td><td>-13</td><td>23</td><td>899</td><td>17315</td><td>35606</td></tr></table></td><td>RGPV DEC 2010</td><td>7</td></tr>
</table>

Numerical differentiation

NUMERICAL DIFFERENTIATION

→ **NEWTON'S FORWARD DIFFERENCE FORMULA TO GET THE DERIVATIVE** -

By Newton's forward difference interpolation formula.

$$f(x) = f(a+ph) = f(a) + p\Delta f(a) + \frac{p(p-1)}{2!}\Delta^2 f(a) + \frac{p(p-1)(p-2)}{3!}\Delta^3 f(a) + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 f(a) + \frac{p(p-1)(p-2)(p-3)(p-4)}{5!}\Delta^5 f(a) + \dots \quad (1)$$

where $p = \frac{x-a}{h}$

Differentiating (1) w.r.t to p, we get

$$f'(x) = f'(a+ph)(h) = \Delta f(a) + \frac{2p-1}{2!}\Delta^2 f(a) + \frac{3p^2-6p+2}{3!}\Delta^3 f(a) + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 f(a) + \frac{5p^4-40p^3+105p^2-100p+24}{5!}\Delta^5 f(a) + \dots$$

$$\boxed{\Rightarrow f'(x) = f'(a+ph) = \frac{1}{h}\left[\Delta f(a) + \frac{2p-1}{2}\Delta^2 f(a) + \frac{3p^2-6p+2}{6}\Delta^3 f(a) + \frac{4p^3-18p^2+22p-6}{4!}\Delta^4 f(a) + \frac{5p^4-40p^3+105p^2-100p+24}{5!}\Delta^5 f(a)\right]} \quad \dots\dots (2)$$

Again differentiating (2) w.r.t. 'p', we get

$$f''(a+ph)h = \frac{1}{h}\left[\Delta^2 f(a) + (p-1)\Delta^3 f(a) + \frac{12p^2-36p+22}{4!}\Delta^4 f(a) + \frac{2p^3-12p^2+21p-}{12}\Delta^5 f(a) + \dots\right]$$

$$\boxed{f''(x) = f''(a+ph) = \frac{1}{h^2}\left[\Delta^2 f(a) + (p-1)\Delta^3 f(a) + \frac{6p^2-18p+11}{12}\Delta^4 f(a) + \frac{2p^3-12p^2+21p-10}{12}\Delta^5 f(a) + \dots\right]} \quad \dots\dots (3)$$

EQUATING (2) and (3) are the formulate to find out the derivatives.

EXAMPLE: find the first and Secound derivatives of the function f(x) at the point $x = 1.1$:

x	1	1.2	1.4	1.6	1.8	2.00
$f(x)$	0.00	0.1280	0.5440	1.2960	1.4320	4.000

(R.G.P.V. Bhopal III Sem, feb2010)

SOLUTION:

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4	Δ^5
1.0	0.00					
		0.1280				
1.2	0.1280		0.2880			
		0.4160		0.0480		
1.4	0.5440		0.3360		-1	
		0.7520		-0.952		5
1.6	1.2960		-0616		4	
		0.1360		3.048		
1.8	1.4320		2.4320			
		2.5680				
2.00	4.000					

Here $a = 1 \quad h = 0.2,$

$a+ph = 1.1, \quad 1+p(0.2) = 1.1, \quad p = 0.5$

$$f'(a+ph) = \frac{1}{h}\left[\Delta f(a) + \frac{2p-1}{2}\Delta^2 f(a) + \frac{3p^2-6p+2}{6}\Delta^3 f(a) + \frac{4p^3-18p^2+22p-6}{24}\Delta^4 f(a) + \frac{5p^4-40p^3+105p^2-100p+24}{120}\Delta^5 f(a)\right]$$

$$f'(1.1) = \frac{1}{0.2}\left[0.1280 + \frac{1-1}{2}(0.2880) + \frac{3(0.5)^2-6\times 0.5+2}{6}(0.0480) + \frac{4(0.5)^3-18(0.5)^2+22(0.5)-6}{24}(-1) + \frac{5(0.5)^4-40(0.5)^3+105(0.5)^2-100(0.5)+24}{120}(5)\right]$$

$$= \frac{1}{0.2}\left[0.1280+0 + \frac{0.75-3+2}{6}(0.048) - \frac{0.5-4.5+11-6}{24} + \frac{0.3125-5+26.25-50+24}{120}(5)\right]$$

$$= \frac{1}{0.2}[0.1280 - 0.002 - 0.0417 - 0.1849]$$

$$= 5[-0.1006] = -0.5030$$

Hence, first derivative of the function $f(x)$ is -0.5030 when $x = 1.1$.

$$f''(a+ph) = \frac{1}{h^2}\left[\Delta^2 f(a) + \frac{6p-6}{6}\Delta^3 f(a) + \frac{12p^2 - 36p + 22}{24}\Delta^4 f(a) + \frac{2p^3 - 12p^2 + 21p - 10}{12}\Delta^5 f(a)\right]$$

$$f''(1.1) = \frac{1}{(0.2)^2}\left[0.2880 + \frac{0.5-1}{1}(0.0480) + \frac{6(0.5)^2 - 18(0.5) + 11}{12}(-1) + \frac{2(0.5)^3 - 12(0.5)^2 + 21(0.5) - 10}{12}(5)\right]$$

$$= 25\left[0.2880 - 0.5(0.0480) - \frac{3.5}{12} - \frac{2.25}{12}(5)\right]$$

$$= 25[0.288 - 0.024 - 0.2917 - 0.9375]$$

$$= 25[-0.9652]$$

$$= -24.13$$

S.NO	RGPV QUESTIONS	Year	Marks
Q. 1	Find $\frac{dy}{dx}$ at x=1.1 from the following table: X: 1.0, 1.2, 1.4, 1.6, 1.8, 2.0 y: 0, 0.128, 0.544, 1.296, 2.432, 4.000	RGPV JUNE 2014	7
Q. 2	Find $\frac{dy}{dx}$ at x=1.5 from the following table: X: 1.5, 2.0, 2.5, 3.0, 3.5, 4.0 y: 3.375, 7.0, 13.625, 24.0, 38.875, 59.0	RGPV JUNE 2011	7

⇒ NUMERICAL DIFFERENTIATION (BACKWARD DIFFERENCES)

Newton's formula for backward difference is

$$f(a+ph) = f(a) + p\nabla f(a) + \frac{p(p+1)}{2!}\nabla^2 f(a) + \frac{p(p+1)(p+2)}{3!}\nabla^3 f(a) + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 f(a)$$

$$= f(a) + p\nabla f(a) + \frac{p^2+p}{2!}\nabla^2 f(a) + \frac{p^3+3p^2+2p}{3!}\nabla^3 f(a) + \frac{p^4+6p^3+11p^2+6p}{4!}\nabla^4 f(a) + \ldots\ldots$$

Differentiating with respect to p, we get

$$hf'(a+ph) = \nabla f(a) + \frac{2p+1}{2!}\nabla^2 f(a) + \frac{3p^2+6p+2}{3!}\nabla^3 f(a)$$

$$f'(a+ph) = \frac{1}{h}\left[\nabla f(a) + \frac{2p+1}{2}\nabla^2 f(a) + \frac{3p^2+6p+2}{6}\nabla^3 f(a) + \frac{2p^3+9p^2+11p+3}{12}\nabla^4 f(a) + \ldots\right]$$

If we replace x by x_p and a by x_n, then we get

$$\boxed{f'(x) = f'(x_n+ph) = \frac{1}{h}\left[\nabla f(x_n) + \frac{2p+1}{2}\nabla^2 f(x_n) + \frac{3p^2+6p+2}{6}\nabla^3 f(x_n) + \frac{2p^3+9p^2+11p+3}{12}\nabla^4 f(x_n) + \ldots\right.}$$

Again differentiating with respect to p, we get

$$f''(a+ph) = \frac{1}{h^2}\left[\nabla^2 f(a) + (p+1)\nabla^3 f(a) + \frac{6p^2+18p+11}{12}\nabla^4 f(a) + \ldots\right]$$

Again we put x_p for x and x_n for a, we get.

$$\boxed{f''(x_p) = f''(x_n+ph) = \frac{1}{h^2}\left[\nabla^2 f(x_n) + (p+1)\nabla^3 f(x_n) + \frac{6p^2+18p+11}{12}\nabla^4 f(x_n) + \right.}$$

Similarly, $f'''(a+ph) = \frac{1}{h^3}\left[\nabla^3 f(a) + \frac{2p+3}{2}\nabla^4 f(a) + \ldots\right]$

EXAMPLE : Given that

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	7.989	8.403	8.781	9.129	9.451	9.750	10.031

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (i) $x = 1.1$ (ii) $x = 1.6$.

(R.G.P.V., Bhopal, II Sem, Dec 2007)

SOLUTION: The difference table is as under

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
1.0	7.989						
		0.414					
1.1	8.403		−0.036				
		0.378		0.006			
1.2	8.781		−0.030		−0.002		
		0.348		0.004		0.001	
1.3	9.129		−0.026		−0.001		0.002
		0.322		0.003		0.003	
1.4	9.451		−0.023		+0.002		
		0.299		0.005			
1.5	9.750		−0.018				
		0.281					
1.6	10.031						

$a = 1.1, \quad x = 1.1, \quad h = 0.1$

$x = a + ph \Rightarrow 1.1 = 1.1 + p(0.1) = p \Rightarrow 0$

$$\frac{dy}{dx} = \frac{1}{h}\Big[\Delta y + \frac{1}{2!}(2p-1)\Delta^2 y + \frac{1}{3!}(3p^2-6p+2)\Delta^3 y + \frac{(4p^3-18p^2+22p-6)}{4!}\Delta^4 y + \frac{5p^4-40p^3+105p^2-100p+24}{\cancel{2}\,5!}\Delta^5 y + \frac{6p^5-75p^4+340p^3-675p^2+548p-120}{6!}\Delta^6 y + \dots\Big]$$

On putting the value of $p = 0$ in (1), we get

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y - \frac{1}{2}\Delta^2 y + \frac{1}{3}\Delta^3 y - \frac{1}{4}\Delta^4 y + \frac{1}{5}\Delta^5 y - \frac{1}{6}\Delta^6 y + \dots\right] \quad \dots (2)$$

Putting the values of $h, \Delta y, \Delta^2 y, \Delta^3 y, \Delta^4 y, \Delta^5 y$ and $\Delta^6 y$ in (2), we get

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(-0.001) + \frac{1}{5}(0.003) + \dots\right]$$

$$= \frac{1}{0.1}[0.378 + 0.015 + 0.0013 + 0.0003 + 0.0006]$$

$$= \frac{1}{0.1}[0.3952] = 3.952 \quad \text{Ans.}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y + (p-1)\Delta^3 y + \frac{6p^2-18p+11}{12}\Delta^4 y + \frac{2p^3-12p^2+21p-10}{12}\Delta^5 y + \frac{15p^4-150p^3+510p^2-675p+274}{360}\Delta^6 y + \cdots\right]$$

when $p=0$, $\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\Delta^2 y - \Delta^3 y + \frac{11}{12}\Delta^4 y - \frac{5}{6}\Delta^5 y + \frac{137}{180}\Delta^6 y + \ldots\right]$ ..(3

On putting the value of $\Delta^2 y$, $\Delta^3 y$, $\Delta^4 y$, $\Delta^5 y$. etc in (3), we get

$$\left(\frac{d^2y}{dx^2}\right)_{1\cdot1} = \frac{1}{(0\cdot1)^2}\left[-0\cdot030 - 0\cdot004 + \frac{11}{12}(-0\cdot001) - \frac{5}{6}(0\cdot003)\right]$$

$$= \frac{1}{0\cdot01}\left[-0\cdot030 - 0\cdot004 - 0\cdot009 - 0\cdot0025\right] = \frac{1}{0\cdot01}\left[-0\cdot0374\right]$$

$$= -3\cdot74 \quad \text{Ans.}$$

ii) To find out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1\cdot6$ we have to use Newton's backward interpolation formula.

Since 1.6 is the end of the given data

$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y + \frac{2p+1}{2!}\nabla^2 y + \frac{3p^2+6p+2}{3!}\nabla^3 y + \frac{2p^3+9p^2+11p+3}{12}\nabla^4 y + \ldots\right]$$

when $p=0$, $\frac{dy}{dx} = \frac{1}{h}\left[\nabla y + \frac{1}{2}\nabla^2 y + \frac{1}{3}\nabla^3 y + \frac{1}{4}\nabla^4 y + \frac{1}{5}\nabla^5 y + \frac{1}{6}\nabla^6 y\right]$ -- (

On putting the values of h, ∇y, $\nabla^2 y$, $\nabla^3 y$, $\nabla^4 y$, $\nabla^5 y$ in (4), we get

$$\left(\frac{dy}{dx}\right)_{1\cdot1} = \frac{1}{0\cdot1}\left[0\cdot281 + \frac{1}{2}(-0\cdot018) + \frac{1}{3}(0\cdot005) + \frac{1}{4}(0\cdot002) + \frac{1}{5}(0\cdot003) + \frac{1}{6}(0\cdot002\right.$$

$$= \frac{1}{0\cdot1}\left[0\cdot281 - 0\cdot009 + 0\cdot0017 + 0\cdot0005 + 0\cdot0006 + 0\cdot003\right]$$

$$= \frac{1}{0\cdot1}\left[0\cdot2751\right] = 2\cdot751 \qquad \text{Ans.}$$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y + (p+1)\nabla^3 y + \frac{6p^2+18p+11}{12}\nabla^4 y + \ldots\right]$$

when $p=0$, $\left(\frac{d^2y}{dx^2}\right) = \frac{1}{h^2}\left[\nabla^2 y + \nabla^3 y + \frac{11}{12}\nabla^4 y + \frac{5}{6}\nabla^5 y + \frac{137}{180}\nabla^6 y\right]$

$$= \frac{1}{(0\cdot1)^2}\left[-0\cdot018 + 0\cdot005 + \frac{11}{12}(0\cdot002) + \frac{5}{6}(0\cdot003) + \frac{137}{180}(0\cdot002)\right]$$

$$= \frac{1}{(0\cdot01)}\left[-0\cdot018 + 0\cdot005 + 0\cdot0018 + 0\cdot0025 + 0\cdot0015\right]$$

$$= \frac{1}{(0\cdot01)}\left[-0\cdot0072\right] = -0\cdot72 \qquad \text{Ans}$$

→ STIRLING FORMULA FOR DERIVATIVE

By stirling formula, we have

$$f(a+ph) = f(a)+p\left[\frac{\Delta f(a)+\Delta f(a-h)}{2}\right]+\frac{p^2}{2!}\Delta^2 f(a-h)$$

$$+\frac{p(p^2-1)}{3!}\left[\frac{\Delta^3 f(a-h)+\Delta^3 f(a-2h)}{2}\right]+\frac{p^2(p^2-1)}{4!}\Delta^4 f(a-2h)+\dots \quad (1)$$

Differentiating (1) w.r.t. 'p', we get

$$hf'(a+ph) = \left[\frac{\Delta f(a)+\Delta f(a-h)}{2}\right]+p\Delta^2 f(a-h)+\frac{3p^2-1}{6}\left[\frac{\Delta^3 f(a-h)+\Delta^3 f(a-2h)}{2}\right.$$

$$+\frac{4p^3-2p}{24}\Delta^4 f(a-2h)+\dots$$

$$\Rightarrow f'(a+ph) = \frac{1}{h}\left[\left\{\frac{\Delta f(a)+\Delta f(a-h)}{2}\right\}+p\Delta^2 f(a-h)\right.$$

$$+\frac{3p^2-1}{6}\left\{\frac{\Delta^3 f(a-h)+\Delta^3 f(a-2h)}{2}\right\}+\frac{2p^3-p}{12}\Delta^4 f(a-2h)+\dots$$

$$\boxed{f''(a+ph) = \frac{1}{h^2}\left[\Delta^2 f(a-h)+p\frac{\{\Delta^3 f(a-h)+\Delta^3 f(a-2h)\}}{2}+\frac{6p^2-1}{2}\{\Delta^4 f(a-2h)\}+\dots\right]}$$

S.NO	RGPV QUESTIONS	Year	Marks
Q.1	Find f′(x) and f″(x) at x=6 given that X: 4.5, 5.0, 5.5, 6.0, 6.5, 7.0, 7.5 F(x): 9.69, 12.9., 16.71, 21.18, 26.37, 32.34, 39.15	RGPV,DEC. 2014	7
Q. 2	A sloder in a machine moves along a fixed straight rod. Its distance x cm. Along the rod is given below for various values of the time t-second . find the velocity of the slider and its acceleration when t=0.3 second. t: 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6 x: 30.13, 31.62, 32.87, 33.64, 33.95, 33.81, 33.24	RGPV,DEC 2013	7
Q. 3	A rod is rotating in a plane. the following table gives the angle ϴ (radians) through which the rod has turned for various values of the time t second: t: 0.0, 0.2, 0.4, 0.6, 0.8, 1.0, 1.2 ϴ: 0.00, 0.12, 0.49, 1.12, 2.02, 3.20, 4.67 Calculate the angular velocity and acceleration of the rod when t=0.6 second	RGPV JUNE 2010	7

Numerical integration

NUMERICAL INTEGRATION

→ TRAPEZOIDAL RULE

$$\boxed{\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+y_3+\dots+y_{n-1})\right]}$$

→ SIMPSON'S ONE THIRD RULE

$$\int_{x_0}^{x_0+nh} f(x)dx = \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+y_6+\dots y_{n-2})+4(y_1+y_3+y_5+\dots y_{n-1})\right]$$

$$\boxed{\int f(x)dx = \frac{h}{3}\left[(y_0+y_n)+2\Sigma y_e+4\Sigma y_o\right]}$$

This is known as the Simpson's one-third rule. It is mostly called simply Simpson's rule.

→ SIMPSON'S THREE-EIGHT RULE

$$\boxed{\int_{x_0}^{x_0+nh} f(x)dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+\dots y_{n-1})+2(y_3+y_6+y_9+\dots y_{n-3})\right]}$$

→ WEDDLES RULE

$$\boxed{\begin{aligned}\int f(x)dx = \frac{3h}{10}&\{(y_0+5y_1)+(y_2+6y_3)+(y_4+5y_5)+y_6\}\\ &+\{(y_6+5y_7)+(y_8+6y_9)+(y_{10}+5y_{11})+y_{12}\}\\ &+\{(y_{12}+5y_{13})+(y_{14}+6y_{15})+(y_{16}+5y_{17})+y_{18}\}\end{aligned}}$$

Note 1. The interval AB must be divided into the multiple of 6 Sub-divisio[n]
2. Weddle's Rule is more accurate than Simpson's rules.
3. The coefficients of each group on R.H.S is 1, 5, 1, 6, 1, 5.

EXAMPLE : Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's $\left(\frac{1}{3}\right)$ rule. Hence obtain the appropriate value of π dividing the range into 6 equal parts.

(R.G.P.V, Bhopal, III Sem, Dec 2001)

SOLUTION :- We divide the range of integration into 6 equal parts by taking $h = \frac{1-0}{6} = \frac{1}{6}$.

Now, the value of given function $y = \left\{\frac{1}{1+x^2}\right\}$ is given as below for each point of sub-division.

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$\frac{1}{1+x^2}$	1,00,000	$\frac{36}{37} = 0.97297$	$\frac{36}{40} = 0.90000$	$\frac{36}{45} = 0.80000$	$\frac{36}{52} = 0.69231$	$\frac{36}{61} = 0.59016$	$\frac{1}{2} = 0.50000$
y	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's one third Rule

$$\int_{x_0}^{x_0+nh} y\,dx = \frac{h}{3}\left[(y_0+y_n) + 2(y_2+y_4+\ldots) + 4(y_1+y_3+\ldots)\right]$$

$$\int_0^1 \frac{1}{1+x^2}dx = \frac{6}{3}\left[(1.0000+0.50000) + 2(0.90000+0.69231) + 4(0.97297+0.80000+0.59016)\right]$$

$$= \frac{1}{18}\left[1.50000 + 2(1.59231) + 4(2.36313)\right]$$

$$= \frac{1}{18}\left[1.50000 + 3.18462 + 9.45252\right]$$

$$= \frac{1}{18}(14.13714) = 0.78539667 \qquad \ldots (1)$$

Again $\int_0^1 \frac{1}{1+x^2}dx = (\tan^{-1}x)_0^1 = (\tan^{-1}1) = \frac{\pi}{4} \qquad \ldots (2)$

from (1) and (2), we have

$$\frac{\pi}{4} = 0.78539667 \Rightarrow \pi = 3.14158668$$

Hence, value of $\pi = 3.14158668$ approximately

EXAMPLE. A river is 80 metres wide. The depth d (in metres) of the river at a distances x from the bank is given by the following table:

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3
y	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

find approximately the area of cross-section of the river using Simpson's $\left(\frac{3}{8}\right)$ rule.

(RGPV Bhopal, June 2010)

SOLUTION: By Simpson's $\left(\frac{3}{8}\right)$ rule, the area of cross-section of the river $= \int_0^{80} f(x)\,dx$

$$= \int_0^{80} y\,dx = \frac{3h}{8}[\{y_0+y_8\}+3\{y_1+y_2+y_4+y_5+y_7\}+2\{y_3+y_6\}]$$

$$= \frac{3}{8}\times 10[(0+3)+3(4+7+12+15+8)+2(9+14)]$$

[Here $y = d$, $h = 10$]

$$= \frac{30}{8}[3+138+146]$$

$$= \frac{30}{8}(187)$$

$= 701.25$ Square metres

$= 701$ Square metres (approx)

Hence, the area of Cross-Section of the river is 701 sq. metres (approx.)

EXAMPLE: Evaluate $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's $\frac{3}{8}$ ~~km~~ rule.

Hence obtain the approximate value of π, dividing the range into 6 equal parts.

(RGPV, Bhopal III Sem Dec. 2011)

SOLUTION: Here, $h = \frac{1-0}{6} = \frac{1}{6}$

x	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
$\frac{1}{1+x^2}$	1.00000	$\frac{36}{37} = 0.97297$	$\frac{36}{40} = 0.90000$	$\frac{36}{45} = 0.80000$	$\frac{36}{52} = 0.69231$	$\frac{36}{61} = 0.59016$	$\frac{1}{2} = 0.50000$
y	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's $\frac{3}{8}$ Rule,

$$\int y\,dx = \frac{3h}{8}\left[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\dots+y_{n-1})+2(y_3+y_6+\dots y_{n-3})\right]$$

$$= \frac{3\left(\frac{1}{6}\right)}{8}\left[1.0000+0.50000+3(0.97297+0.90000+0.69231+0.59016)+2(0.8000)\right]$$

$$= \frac{1}{16}\left[1.50000+3(3.15544)+2(0.80000)\right]$$

$$= \frac{1}{16}\left[1.50000+9.46632+1.60000\right]$$

$$= \frac{1}{16}\left[12.56632\right]$$

$$= 0.785395 \qquad \dots (1)$$

Again $\int_0^1 \frac{dx}{1+x^2} = (\tan^{-1}x)_0^1 = (\tan^{-1}1) = \frac{\pi}{4} \qquad \dots (2)$

from (1) and (2), we have

$$\frac{\pi}{4} = 0.785395 \Rightarrow \pi = 3.14158 \qquad \text{Ans}$$

Hence the approximate value of π is 3.14158.

S.NO	RGPV QUESTIONS	Year	Marks
Q.1	Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using Weddle's rule. Hence obtain the approximate value of Π.	RGPV,DEC 2014, JUNE 2013	7
Q. 2	Evaluate the integral $\int_0^{0.6} e^{-x^2} dx$ by Simpson $\frac{1}{3}$ rule	RGPV,JUNE 2014	3
Q. 3	Calculate the value of $\int_0^{\Pi/2} \sin x dx$ by Simpson $\frac{1}{3}$ rule using II ordinates.	RGPV,DEC. 2013	7
Q. 4	Evaluate $\int_0^{\Pi/2} \sqrt{\cos x} dx$ (i) Using Simpson $\frac{1}{3}$ rule (ii) Using Weddle's rule.	RGPV,JUNE 2011	7
Q. 5	A river is 80 ft. Wide. the depth d in feet at a distance x ft from one bank is given below by the following table: X: 0, 10, 20, 30, 40, 50, 60, 70, 80 Y: 0, 4, 7, 9, 12, 15, 14, 8, 3 Find approximately the area of the cross section.	RGPV,JUNE 2010	7

Reference

Book	Author
Higher Engg. Mathematics	B.S.Grewal
Engg. Mathematics - III	Dr. D.C.Agarwal
Engg. Mathematics - III	H . K. DASS