Unit-1
Fourier series

Syllabus:
Fourier Series: Introduction of Fourier series , Fourier series for Discontinuous functions, Fourier series for even and odd function, Half range series.

Fourier series

Introduction

A Taylor series is an infinite series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

where $a_0, a_1, a_2, \ldots$ are constants, called the **coefficients** of the series. A Taylor series does not include terms with negative powers. A Fourier series is an infinite series expansion in terms of trigonometric functions

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

Any piecewise smooth function defined on a finite interval has a Fourier series expansion.

Periodic Functions

A function satisfying the identity $f(x) = f(x + T)$ for all x, where $T > 0$, is called periodic or T-periodic as shown in figure:

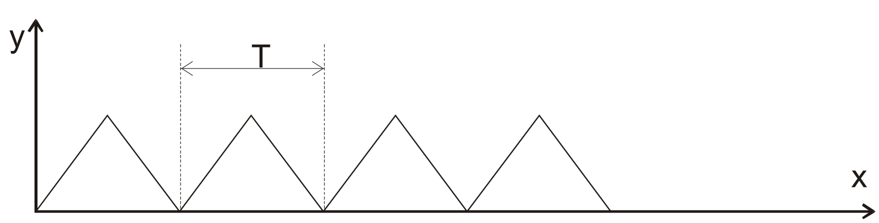


Figure A T-periodic function.

For a T-periodic function

$$f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \ldots = f(x + nT)$$

If T is a period then nT is also a period for any integer $n > 0$. T is called a fundamental period. The definite integral of a T-periodic function is the same over any interval of length T. Example 2.1-1 will use this property to integrate a 2-periodic function shown in figure.

Example.1: Let f be the 2-periodic function and N is a positive integer. Compute $\int_{-N}^{N} f^2(x)dx$ if $f(x) = -x + 1$ on the interval $0 \le x \le 2$

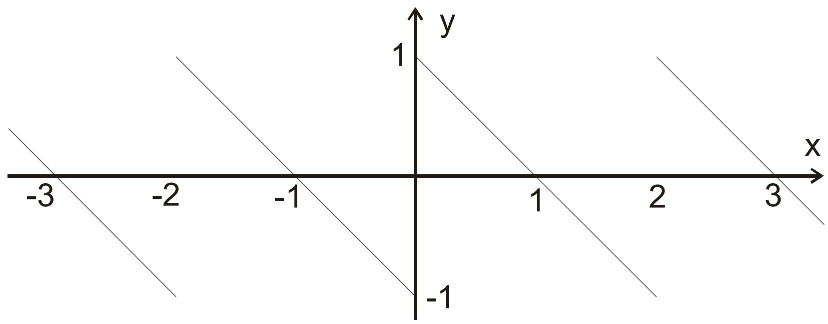


Figure A 2-periodic function.

Solution

$$\int_{-N}^{N} f^2(x)dx = \int_{-N}^{-N+2} f^2(x)dx + \int_{-N+2}^{-N+4} f^2(x)dx + \ldots + \int_{N-2}^{N} f^2(x)dx$$

$$\int_{-N}^{N} f^2(x)dx = N\int_{N-2}^{N} f^2(x)dx = N\int_{0}^{2} (-x+1)^2 dx = N\left\{-\frac{1}{3}(-x+1)^3\right\}\Bigg|_0^2$$

$$\int_{-N}^{N} f^2(x)dx = -\frac{N}{3}[-1-1] = \frac{2}{3}N$$

The most important periodic functions are those in the (2π-period) trigonometric system

1, cos x, cos $2x$, cos $3x$, …, cos mx, …,
sin x, sin $2x$, sin $3x$, …, sin nx, …,

Orthogonal functions

If $\int_a^b f(x)g(x)dx = 0$ then f and g are orthogonal over the interval [a, b].

Examples of orthogonal functions:

$$\int_{-\pi}^{\pi} \cos mx \cos nx dx = 0 \text{ for } m \neq n$$
$$= \pi \text{ for } m = n$$

$\int_{-\pi}^{\pi} \sin mx \sin nx dx$ = 0 for $m \neq n$

= π for $m = n$

$\int_{-\pi}^{\pi} \cos mx \sin nx dx$ = 0 for all m and n

Fourier series are special expansions of functions of the form

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos(nx) + b_n \sin(nx))$$

where the coefficients $a_0, a_1, a_2, \ldots, b_1, b_2, \ldots$ must be evaluated.

The coefficient a_0 is determined by integrating both sides of Eq. (2.1-1) over the interval $[-\pi, \pi]$.

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} (a_n \cos(nx) + b_n \sin(nx))dx$$

Since $\int_{-\pi}^{\pi} \cos nx dx = \int_{-\pi}^{\pi} \sin nx dx = 0$ for $n = 1, 2, \ldots$

$$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 dx = 2\pi a_0 \Rightarrow a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

The coefficient a_n is determined by multiplying both sides of Eq. with cos mx and integrating the resulting equation over the interval $[-\pi, \pi]$.

$$\int_{-\pi}^{\pi} f(x)\cos(mx)dx = \int_{-\pi}^{\pi} a_0 \cos(mx)dx + \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n \cos(nx)\cos(mx)dx$$

$$+ \sum_{n=1}^{\infty} \int_{-\pi}^{\pi} b_n \sin(nx)\cos(mx)dx$$

Since $\int_{-\pi}^{\pi} \cos mx dx = 0$, $\int_{-\pi}^{\pi} \cos mx \sin nx dx = 0$ for all m and $\int_{-\pi}^{\pi} \cos mx \cos nx dx = 0$ for $m \neq n$

$$\int_{-\pi}^{\pi} f(x)\cos(mx)dx = a_n \int_{-\pi}^{\pi} (\cos nx)^2 dx = \pi a_n$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)dx = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\cos(mx)dx$$

Similarly the coefficient b_n is determined by multiplying both sides of Eq. with sin mx and integrating the resulting equation over the interval [–π, π].

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)dx = \frac{1}{\pi}\int_{0}^{2\pi} f(x)\sin(mx)dx$$

Example.2: Solve the one dimensional heat equation with no heat generation, zero boundary conditions (0°C), and constant initial temperature of 100°C.

Solution

The partial differential equation for one-dimensional heat conduction is

$$\rho C_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} + Q(x, t)$$

Since there is no heat generation $Q(x, t) = 0$

$$\rho C_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2} \Rightarrow \frac{\partial T}{\partial t} = \alpha\frac{\partial^2 T}{\partial x^2} \text{ where } \alpha = \frac{k}{\rho C_p}$$

The boundary and initial conditions are

$$T(0, t) = T(L, t) = 0^\circ C; \; T(x, 0) = 100^\circ C.$$

The solution for the temperature is

$$T(x, t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\exp\left[-\left(\frac{n\pi}{L}\right)^2 \alpha t\right]$$

At $t = 0$, $T(x, 0) = 100 = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$

This is the Fourier sine series (with $a_0, a_1, a_2, \ldots = 0$) where the coefficients $b_1, b_2, \ldots$ can be determined by by multiplying both sides of the above equation with sin mx and integrating the resulting equation over the interval [0, L].

$$100\int_0^L \sin\frac{m\pi x}{L}dx = \sum_{n=1}^{\infty} b_n\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx$$

Since $\int_0^L \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx = 0$ for $m \neq n$

$$100\int_0^L \sin\frac{n\pi x}{L}\,dx = b_n\int_0^L\left(\sin\frac{n\pi x}{L}\right)^2 dx$$

Using the identity $\sin^2 x = \frac{1-\cos 2x}{2}$, the above equation becomes

$$100\,\frac{L}{n\pi}\left[-\cos\frac{n\pi x}{L}\right]_0^L = b_n\int_0^L \frac{1}{2}\left(1-\cos\frac{2n\pi x}{L}\right)dx$$

$$100\,\frac{L}{n\pi}[-\cos n\pi + 1] = \frac{b_n}{2}\left[x-\frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\right]_0^L = \frac{b_n}{2}L$$

$$b_n = \frac{200}{n\pi}[1-\cos n\pi]$$

For n = even, $\cos(n\pi) = 1 \Rightarrow b_n = 0$

For n = odd, $\cos(n\pi) = -1 \Rightarrow b_n = \frac{400}{n\pi}$

The Fourier expansion for 100 is then given by

$$f(x) = 100 = \sum_{n=1}^{\infty} b_{2n-1}\sin\frac{(2n-1)\pi x}{L} \text{ where } b_{2n-1} = \frac{400}{(2n-1)\pi}$$

The plot of $f(x) = \sum_{n=1}^{51} b_{2n-1}\sin\frac{(2n-1)\pi x}{L}$ for 51 terms is a good approximation of 100 away from the end points as shown in Figure 2.1-1. There is a 18 % overshoot called Gibbs phenomenon near the end points. Gibbs phenomenon occurs only when a finite series of eigenfunctions approximates a discontinuous function.

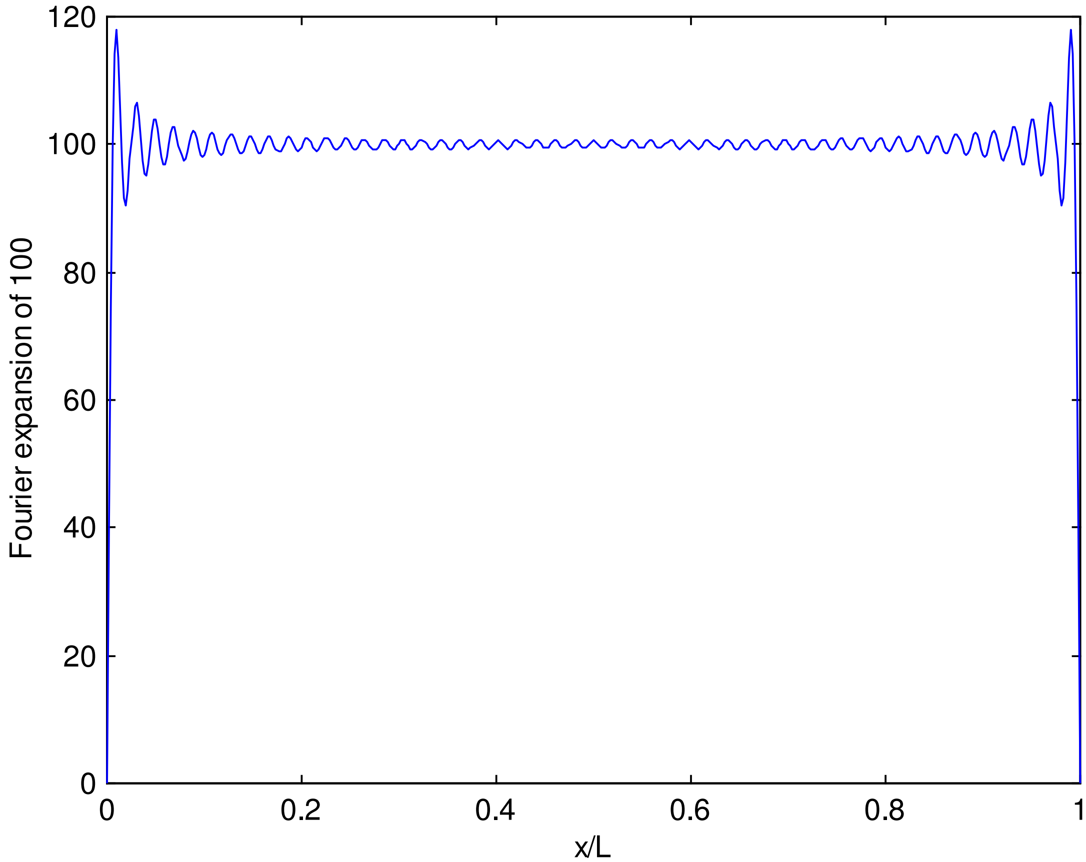


Figure. points approximation of $f(x)$ = 100 = $\sum_{n=1}^{\infty} b_{2n-1} \sin \frac{(2n-1)\pi x}{L}$

Definition 1 (Periodic functions)
A function $f(t)$ is said to have a **period** T or to be **periodic** with period T if for all t, $f(t+T)=f(t)$, where T is a positive constant. The least value of $T>0$ is called the **principal period** or the **fundamental period** or simply the **period** of $f(t)$.

Example3.

The function $\sin x$ has periods $2\pi, 4\pi, 6\pi, \ldots$, since $\sin(x+2\pi), \sin(x+4\pi), \sin(x+6\pi), \ldots$ all equal $\sin x$.

Example4.

Let $a \in \mathbb{R}$. If f(x) has the period 2π then $F(t) := f(\omega t) := f(\frac{2\pi}{T}t)$ has the period T. (substitute $\frac{2\pi}{T}t := x, \quad \omega := \frac{2\pi}{T}$)

Example5.

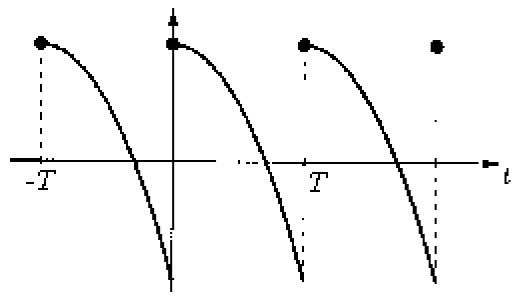
If f has the period T then

$$\int_a^{a+T} f(t)dt = \int_0^T f(t)dt \quad \forall a \in \mathbb{R}$$

Definition (Periodic expansion)

Let a function *f* be declared on the interval [0,*T*). The **periodic expansion** $\tilde{f}$ of *f* is defined by the formula

$$\tilde{f}(t) = \begin{cases} f(t) & 0 \le t < T \\ \tilde{f}(t-T) & \forall t \in \mathbb{R} \end{cases}$$



Definition (Piecewise continuous functions)

A function f defined on *I*=[*a*,*b*] is said to be **piecewise continuous** on *I* if and only if

(i)

there is a subdivision $a = x_0 < x_1 < x_2 < \ldots < x_n = b$ such that *f* is continuous on each subinterval $I_k = \{x : x_{k-1} < x < x_k\}$ and

(ii)

at each of the subdivision points $x_0, x_1, \ldots, x_n$ both one-sided limits of *f* exist.

Theorem

Let *f* be continuous on $I = [-\pi, \pi]$. Suppose that the series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\ nx + b_n \sin nx) \tag{1}$$

converges uniformly to *f* for all $x \in I$. Then

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos ntdt \quad n = 0, 1, 2, \ldots \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin ntdt \quad n = 1, 2, \ldots \end{aligned} \tag{2}$$

Definition (Fourier coefficients, Fourier series)

The numbers a_n and b_n are called the **Fourier coefficients** of f. When a_n and b_n are given by (2), the trigonometric series (1) is called the **Fourier series** of the function f.

Remark

If f is any integrable function then the coefficients a_n and b_n may be computed. However, there is no assurance that the Fourier series will converge to f if f is an arbitrary integrable function. In general, we write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos\ nx + b_n \sin nx)$$

to indicate that the series on the right may or may not converge to f at some points.

Remark (Complex Notation for Fourier series)

Using Euler's identities,

$$e^{i\theta} = \cos\theta + i\sin\theta$$

where i is the imaginary unit such that i^2=-1, the Fourier series of $f(x)$ can be written in complex form as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} \tag{3}$$

where

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx \tag{4}$$

and

$$c_0 = \frac{1}{2}a_0, \quad c_n = \frac{1}{2}(a_n - ib_n), \quad c_{-n} = \frac{1}{2}(a_n + ib_n), \qquad n = 1, 2, \ldots$$

$$a_0 = 2c_0, \quad a_n = c_n + c_{-n}, \quad b_n = i(c_n - c_{-n}), \qquad n = 1, 2, \ldots$$

Example.6:

Let $f(x)$ be defined in the interval $[0,T]$ and determined outside of this interval by its periodic extension, i.e. assume that $f(x)$ has the period T. The Fourier series corresponding to $f(x)$ (with $\omega := \frac{2\pi}{T}$) is

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega x + b_n \sin n\omega x) \tag{5}$$

where the Fourier coefficients a_n and b_n are

$$a_n = \frac{2}{T}\int_0^T f(x)\cos n\omega x dx \qquad n = 0, 1, 2, \ldots \tag{6}$$

$$b_n = \frac{2}{T}\int_0^T f(x)\sin n\omega x dx \qquad n = 1, 2, \ldots \tag{7}$$

Example.7:

Let a_n and b_n be the Fourier coefficients of f. The **phase angle form** of the Fourier series of f is

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega x + \delta_n)$$

with

$$c_n = \sqrt{a_n^2 + b_n^2} \qquad n = 1, 2, \ldots$$

and

$$\delta_n = \tan^{-1}(-\frac{b_n}{a_n}), \qquad n = 1, 2, \ldots$$

Example.8:

We compute the Fourier series of the function f given by

$$f(x) = \begin{cases} 1 & , 0 \le x < \pi \\ -1 & , \pi \le x < 2\pi \end{cases}$$

Since f is an odd function, so is $f(x)\cos nx$, and therefore

$$a_n = 0, \qquad n = 1, 2, 3, \ldots$$

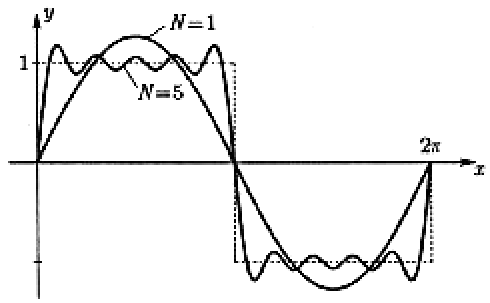
a_0=0

For $n \geq 1$ the coefficient b_n is given by

$$b_n = \frac{1}{\pi}\left(\int_0^{\pi} \sin nx dx - \int_{\pi}^{2\pi} \sin nx\right) = \begin{cases} \frac{4}{n\pi} & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

It follows

$$f \sim \frac{4}{\pi}(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots)$$



Dirichlet conditions

It is important to establish simple criteria which determine when a Fourier series converges. In this section we will develop conditions on $f(x)$ that enable us to determine the sum of the Fourier series. One quite useful method to analyse the convergence properties is to express the partial sums of a Fourier series as integrals. Riemann and Fejer have since provided other ways of summing Fourier series. In this section we limit the study of convergence to functions that are piecewise smooth on a given interval.

Definition (Piecewise smooth function)

A function f is **piecewise smooth** on an interval if both f and f' are piecewise continuous on the interval.

Theorem

Suppose that f is piecewise smooth and periodic.

Then the series (1) with coefficients (2) converges to

1. $f(x)$ if x is a point of continuity.
2. $\frac{1}{2}(f(x+0) + f(x-0))$ if x is a point of discontinuity.

This means that, at each x between $-L$ and L, the Fourier series converges to the average of the left and the right limits of $f(x)$ at x. If $f$is continuous at x, then the left and the right limits are both equal to $f(x)$, and the Fourier series converges to $f(x)$ itself. If f has a jump discontinuity at $x$then the Fourier series converges to the point midway in the gap at this point.

Remark

Let f be a given piecewise continuous function. We say that f is **standardised** if its values at points x_i of discontinuity are given by

$$f(x_i) = \frac{1}{2}[f(x_i+) + f(x_i-)]$$

Remark

The conditions imposed on $f(x)$ are sufficient but not necessary, i.e if the conditions are satisfied the convergence is guaranteed. However, if they are not satisfied the series may or may not converge.

Theorem (Bessel's inequality)

Suppose that f is integrable on the interval $[0,T]$. Let a_n, b_n, c_n be the Fourier coefficients of f. Then

$$\frac{|a_0|^2}{2} + \sum_{n=1}^{\infty}(|a_n|^2 + |b_n|^2) = 2\sum_{k=-\infty}^{\infty}|c_k|^2 \le \frac{2}{T}\int_0^T |f(t)|^2 dt \tag{8}$$

Theorem (Riemann lemma)

Let f be integrable and a_n and b_n be the Fourier coefficients of f. Then

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$$

which means

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\cos nt dt = \lim_{n\to\infty}\int_{-\pi}^{\pi} f(t)\sin nt dt = 0$$

Theorem (Parseval's identity)

$$\frac{2}{T}\int_0^T |f(x)|^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) \tag{9}$$

if a_n and b_n are the Fourier coefficients corresponding to $f(x)$ and if $f(x)$satisfies the Dirichlet conditions.

The Gibbs Phenomenon

Near a point, where f has a jump discontinuity, the partial sums S_n of a Fourier series exhibit a substantial overshoot near these endpoints, and an increase in n will not diminish the amplitude of the overshoot, although with increasing n the the overshoot occurs over smaller and smaller intervals. This phenomenon is called Gibbs phenomenon. In this section we examine some detail in the behaviour of the partial sums S_n of $S(x) = \sum_{k=1}^{\infty} \frac{\sin kx}{k}$.

Theorem

$$\sum_{k=1}^{\infty} \frac{\sin kx}{x} = \frac{\pi - x}{2}, \quad 0 < x < 2\pi$$

The next step is to replace the partial sums S_n with integrals

$$S_n(x) = \int_0^x \sum_{k=1}^{n} \cos(kt)dt = \frac{-x}{2} + \int_0^x \frac{\sin(\frac{2n+1}{2}t)}{2\sin(t/2)}dt \to \frac{\pi - x}{2} \qquad (n \to \infty)$$

For $x \approx 0$ we have a typically "overshoot". This will be the next step to show. Let $x_n = \frac{2\pi}{2n+1}$.

$$S_n(x_n) + \frac{1}{2}x_n = \int_0^{x_n} \frac{\sin(\frac{2n+1}{2}t)}{2\sin(t/2)}dt = \int_0^{\pi} \frac{\sin\tau}{\sin(\frac{\tau}{2n+1})(2n+1)}d\tau \to \int_0^{\pi} \frac{\sin\tau}{\tau}d\tau \qquad (n \to \infty)$$

Theorem (The Gibbs phenomenon)

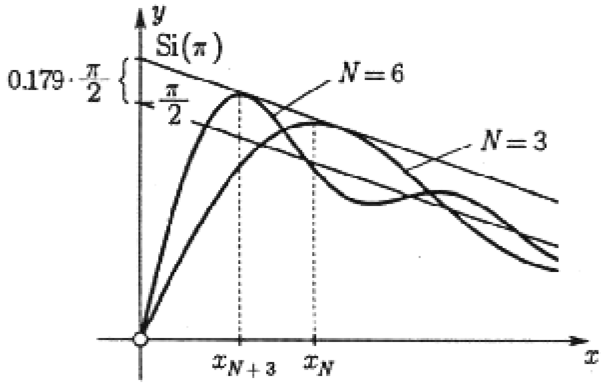
Let $n \in \mathbb{N}$ and $x_n = \frac{2\pi}{2n+1}$.

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{\sin(kx_n)}{k} \right) = \int_0^{\pi} \frac{\sin\tau}{\tau}d\tau$$

and

$$\int_0^{\pi} \frac{sin\tau}{\tau}d\tau = \frac{\pi}{2} \cdot 1.1789797\ldots$$

Since $S(x) \approx \pi/2$ for *x* near 0, we see that an "overshoot" by approximately 17.9% is maintained as $n \to \infty$ (but over smaller and smaller intervals centred at *x*=0).



Often we are interested in properties of a function *f*, knowing only measured values of *f* at equally spaced time intervals

$$t_k = k\Delta t, \quad k \in \mathbb{Z}, \Delta t > 0$$

If this discrete function *f* has the period $T = N\Delta t$, then *f* is described by the vector

$$\mathbf{y} := \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{pmatrix} = \begin{pmatrix} f(0) \\ f(\Delta t) \\ \vdots \\ f((N-1)\Delta t) \end{pmatrix}$$

Definition (Discrete Fourier coefficient)
Assume $T = 2\pi$ then the Fourier coefficient of *y* is defined

$$c_k := \frac{1}{N} \sum_{j=0}^{N-1} y_j e^{-kj\frac{2\pi i}{N}}, \quad k = 0, 1, \ldots, N-1$$

Definition (Discrete Fourier transform (DFT))
The mapping $F : \mathbb{C}^N \to \mathbb{C}^N$, defined by

$$F(y) = c, \quad c = \begin{pmatrix} c_0 \\ \vdots \\ c_{N-1} \end{pmatrix} \in \mathbb{C}^N, \quad y = \begin{pmatrix} y_0 \\ \vdots \\ y_{N-1} \end{pmatrix} \in \mathbb{C}^N$$

with

$$c_k := \frac{1}{N} \sum_{j=0}^{N-1} y_j \overline{w_N^{kj}}, \qquad k = 0, 1, \ldots, N-1 \tag{14}$$

$$w_N := e^{\frac{2\pi i}{N}}$$

is called the **discrete Fourier transform (DFT)**.

If we use the $N \times N$ **Fourier-Matrix**

$$F_n := \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & w_N & w_N^2 & \cdots & w_N^{N-1} \\ 1 & w_N^2 & w_N^4 & \cdots & w_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \\ 1 & w_N^{N-1} & w_N^{2(N-1)} & \cdots & w_N^{(N-1)^2} \end{pmatrix}$$

then we can write (14) as

$$c = \frac{1}{N} F_N y$$

Theorem

It follows, that

$$F_N \overline{F}_N = \overline{F}_N F_N = NE$$

i.e.

$$F_N^{-1} = \frac{1}{N} \overline{F}_N$$

Definition (Inverse discrete Fourier transform (IDFT))

The inverse mapping $y=F_N\, c$ is called the inverse discrete Fourier transform (IDFT)

$$y_j = \sum_{k=0}^{N-1} c_k w_N^{jk}, \qquad j = 0, 1, \ldots, N-1 \tag{15}$$

Some properties of the DFT are:

Linearity

$$\alpha y + \beta z \xrightarrow{\text{DFT}} \alpha c + \beta d$$

Parseval

$$\sum_{k=0}^{N-1} |c_k|^2 = \frac{1}{N} \sum_{k=0}^{N-1} |y_k|^2$$

Theorem (Fast Fourier Transform (FFT))

IF N is even (N=2M), then $y = F_N c$ (and analog $c = \frac{1}{N}\overline{F_N}y$) can be put down to two discrete transforms. We divide c in its odd and even indices

$$e := (c_0, c_2, \dots, c_{N-2})^T \in \mathbb{C}^M$$

and

$$o := (c_1, c_3, \dots, c_{N-1})^T \in \mathbb{C}^M$$

$$y_k = \sum_{j=0}^{N-1} w_N^{jk} c_j = \sum_{j=0}^{M-1} (w_N^2)^{kj} e_j + w^k \sum_{j=0}^{M-1} (w_N^2)^{kj} o_j \quad k = 0, 1, \dots, N-1$$

y is splitted in

$$a := (y_0, y_1, \dots, y_{M-1})^T \in \mathbb{C}^M$$

and

$$b := (y_M, y_{M+1}, \dots, y_{N-1})^T \in \mathbb{C}^M$$

It follows ($w_N^{k+M} = -w_N^k$) that

$$a_k = \sum_{j=0}^{M-1} (w_N^2)^{kj} e_j + w^k \sum_{j=0}^{M-1} (w_N^2)^{kj} o_j \quad k = 0, 1, \dots, M-1$$
$$b_k = \sum_{j=0}^{M-1} (w_N^2)^{kj} e_j - w^k \sum_{j=0}^{M-1} (w_N^2)^{kj} o_j \quad k = 0, 1, \dots, M-1$$

w_N^2 is an Mth root of unity, so the above sums describe two IDFT

$$a = F_M e + \mathrm{Diag}(1, w_M, \dots, w_M^{M-1}) F_M o$$
$$b = F_M e - \mathrm{Diag}(1, w_M, \dots, w_M^{M-1}) F_M o$$

In order to perform a Fourier transform of length *N*, one need to do two Fourier transforms $F_M\, e$ and $F_M\, o$ of length *M* on the even and odd elements. We now have two transforms which take less time to work out. The two sub-transforms can then be combined with the appropriate factor w^k to give the IDFT. Applying this recursively leads to the algorithm of the Fast Fourier transform (FFT).

The Fourier Series is an infinite series expansion involving trigonometric functions.

A periodic waveform $f(t)$ of period $p = 2L$ has a Fourier Series given by:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

$$= \frac{a_0}{2} + a_1\cos\frac{\pi t}{L} + a_2\cos\frac{2\pi t}{L} + a_3\cos\frac{3\pi t}{L} + \ldots$$

$$+ b_1\sin\frac{\pi t}{L} + b_2\sin\frac{2\pi t}{L} + b_3\sin\frac{3\pi t}{L} + \ldots$$

Helpful Revision

Summation Notation (∑)

where

a_n and b_n are the **Fourier coefficients**, and

$\frac{a_0}{2}$ is the **mean value**, sometimes referred to as the **dc level**.

Fourier Coefficients For Full Range Series Over Any Range -*L* TO *L*

If $f(t)$ is expanded in the range $-L$ to L (period = $2L$) so that the range of integration is $2L$, i.e. half the range of integration is L, then the Fourier coefficients are given by

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\,dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt \quad b_n = \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt$$

where $n = 1, 2, 3 \ldots$

NOTE: Some textbooks use

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(t)dt$$

and then modify the series appropriately. It gives us the same final result.

Dirichlet Conditions

Any periodic waveform of period $p = 2L$, can be expressed in a Fourier series provided that

(a) it has a finite number of discontinuities within the period $2L$;

(b) it has a finite average value in the period $2L$;

(c) it has a finite number of positive and negative maxima and minima.

When these conditions, called the Dirichlet conditions, are satisfied, the Fourier series for the function $f(t)$ exists.

Each of the examples in this chapter obey the Dirichlet Conditions and so the Fourier Series exists.

Example of a Fourier Series - Square Wave

Sketch the function for 3 cycles:

$$f(t) = \begin{cases} 0 & \text{if } -4 \le t < 0 \\ 5 & \text{if } 0 \le t < 4 \end{cases}$$

$f(t) = f(t + 8)$

Find the Fourier series for the function.

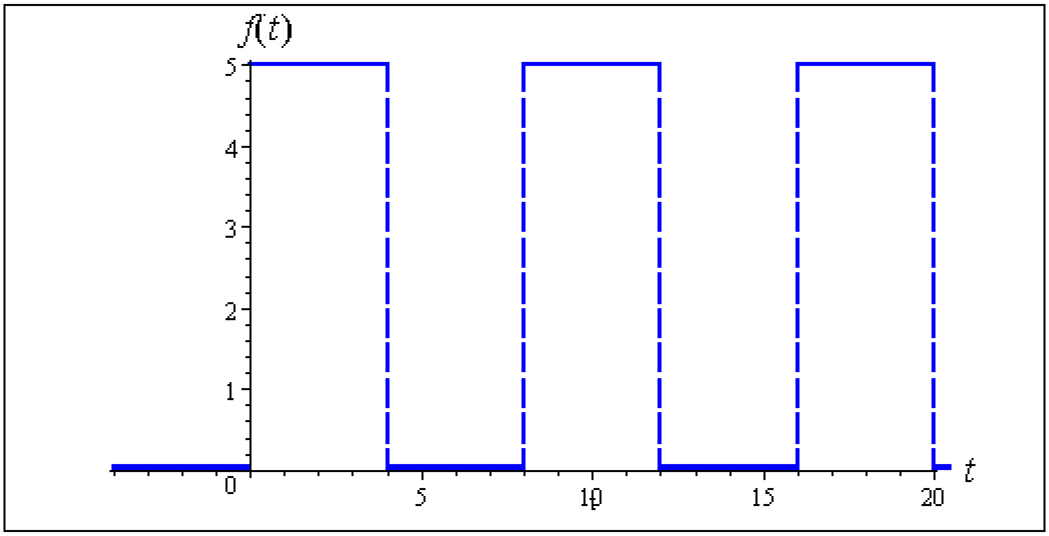
Solution:

First, let's see what we are trying to do by seeing the final answer using a LiveMath animation.

Now for one possible way to solve it:

Answer

The sketch of the function:



We need to find the **Fourier coefficients** a_0, a_n and b_n before we can determine the series.

$$
\begin{aligned}
a_0 &= \frac{1}{L}\int_{-L}^{L} f(t)\,dt \\
&= \frac{1}{4}\int_{-4}^{4} f(t)\,dt \\
&= \frac{1}{4}\left(\int_{-4}^{0} (0)\,dt + \int_{0}^{4} (5)\,dt\right) \\
&= \frac{1}{4}\left(0 + [5t]_0^4\right) \\
&= \frac{1}{4}(20) \\
&= 5
\end{aligned}
$$

Note 1: We could have found this value easily by observing that the graph is totally above the t-axis and finding the area under the curve from $t = 4$ to $t = 4$. It is just 2 rectangles, one with height 0 so the area is 0, and the other rectangle has dimensions 4 by 5, so the area is 20. So the integral part has value 20; and 1/4 of 20 = 5.

Note 2: The mean value of our function is given by $a_0/2$. Our function has value 5 for half of the time and value 0 for the other half, so the value of $a_0/2$ must be 2.5. So a_0 will have value 5.

These points can help us check our work and help us understand what is going on. However, it is good to see how the integration works for a split function like this.

$$
\begin{aligned}
a_n &= \frac{1}{L}\int_{-L}^{L} f(t)\cos\frac{n\pi t}{L}\,dt \\
&= \frac{1}{4}\int_{-4}^{4} f(t)\cos\frac{n\pi t}{4}\,dt \\
&= \frac{1}{4}\left(\int_{-4}^{0}(0)\cos\frac{n\pi t}{4}\,dt + \int_{0}^{4}(5)\cos\frac{n\pi t}{4}\,dt\right) \\
&= \frac{1}{4}\left(0 + \left[5\,\frac{4}{n\pi}\sin\frac{n\pi t}{4}\right]_0^4\right) \\
&= \frac{1}{4}\times 5\times\frac{4}{n\pi}\left(\left[\sin\frac{n\pi t}{4}\right]_0^4\right) \\
&= \frac{5}{n\pi}\left(\sin\frac{n\pi(4)}{4} - \sin\frac{n\pi(0)}{4}\right) \\
&= \frac{5}{n\pi}(\sin n\pi - 0) \\
&= 0
\end{aligned}
$$

Note: In the next section, Even and Odd Functions, we'll see that we don't even need to calculate a_n in this example. We can tell it will have value 0 before we start.

$$
\begin{aligned}
b_n &= \frac{1}{L}\int_{-L}^{L} f(t)\sin\frac{n\pi t}{L}\,dt \\
&= \frac{1}{4}\int_{-4}^{4} f(t)\sin\frac{n\pi t}{4}\,dt \\
&= \frac{1}{4}\left(\int_{-4}^{0}(0)\sin\frac{n\pi t}{4}\,dt + \int_{0}^{4}(5)\sin\frac{n\pi t}{4}\,dt\right) \\
&= \frac{1}{4}\left(0 + \left[-5\,\frac{4}{n\pi}\cos\frac{n\pi t}{4}\right]_0^4\right) \\
&= -\frac{1}{4}\times 5\times\frac{4}{n\pi}\left[\cos\frac{n\pi t}{4}\right]_0^4 \\
&= -\frac{5}{n\pi}\left(\cos\frac{n\pi(4)}{4} - 1\right) \\
&= -\frac{5}{n\pi}\left(\cos n\pi - 1\right)
\end{aligned}
$$

At this point, we can substitute this into our Fourier Series formula:

$$
\begin{aligned}
f(t) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n\sin\frac{n\pi t}{L} \\
&= \frac{5}{2} + \sum_{n=1}^{\infty}(0)\cos\frac{n\pi t}{4} + \sum_{n=1}^{\infty} -\frac{5}{n\pi}\left(\cos n\pi - 1\right)\sin\frac{n\pi t}{4} \\
&= 2.5 - \frac{5}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(\cos n\pi - 1\right)\sin\frac{n\pi t}{4}
\end{aligned}
$$

Now, we substitute $n = 1, 2, 3, \ldots$ into the expression inside the series:

n	$\frac{1}{n}(\cos n\pi - 1)\sin\frac{n\pi t}{4}$
1	$\frac{1}{1}(\cos\pi - 1)\sin\frac{\pi t}{4} = -2\sin\frac{\pi t}{4}$
2	$\frac{1}{2}(\cos 2\pi - 1)\sin\frac{2\pi t}{4} = 0$
3	$\frac{1}{3}(\cos 3\pi - 1)\sin\frac{3\pi t}{4} = -\frac{2}{3}\sin\frac{3\pi t}{4}$
4	$\frac{1}{4}(\cos 4\pi - 1)\sin\frac{4\pi t}{4} = 0$
5	$\frac{1}{5}(\cos 5\pi - 1)\sin\frac{5\pi t}{4} = -\frac{2}{5}\sin\frac{5\pi t}{4}$
6	$\frac{1}{6}(\cos 6\pi - 1)\sin\frac{6\pi t}{4} = 0$
7	$\frac{1}{7}(\cos 7\pi - 1)\sin\frac{7\pi t}{4} = -\frac{2}{7}\sin\frac{7\pi t}{4}$

Now we can write out the first few terms of the required Fourier Series:

$$f(t) = 2.5 - \frac{5}{\pi}\left(-2\sin\frac{\pi t}{4} - \frac{2}{3}\sin\frac{3\pi t}{4} - \frac{2}{5}\sin\frac{5\pi t}{4} - \ldots\right)$$
$$= 2.5 + \frac{10}{\pi}\left(\sin\frac{\pi t}{4} + \frac{1}{3}\sin\frac{3\pi t}{4} + \frac{1}{5}\sin\frac{5\pi t}{4} + \ldots\right)$$

Alternatively, we could observe that every even term is 0, so we only need to generate odd terms. We could have expressed the b_n term as:

$$b_n = -\frac{5}{n\pi}(\cos n\pi - 1)$$
$$= -\frac{5}{n\pi}((-1)^n - 1)$$
$$= \frac{10}{n\pi} \qquad n \text{ odd, (0 if } n \text{ is even)}$$

To generate odd numbers for our series, we need to use:

$$b_n = \frac{10}{(2n-1)\pi} \qquad n = 1, 2, 3\ldots$$

We also need to generate only odd numbers for the sine terms in the series, since the even ones will be 0.

So the required series this time is:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

$$= \frac{5}{2} + \sum_{n=1}^{\infty} (0)\cos\frac{n\pi t}{4} + \sum_{n=1}^{\infty} \frac{10}{(2n-1)\pi} \sin\frac{(2n-1)\pi t}{4}$$

$$= 2.5 + \frac{10}{\pi}\sum_{n=1}^{\infty} \frac{1}{(2n-1)} \sin\frac{(2n-1)\pi t}{4}$$

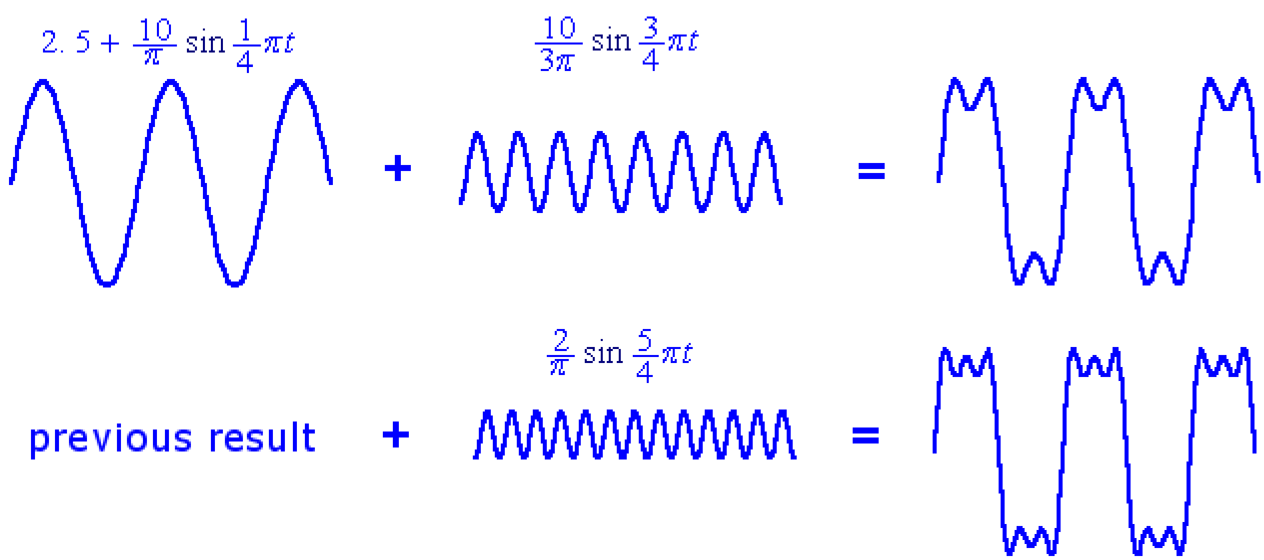
The first four terms series are once again:

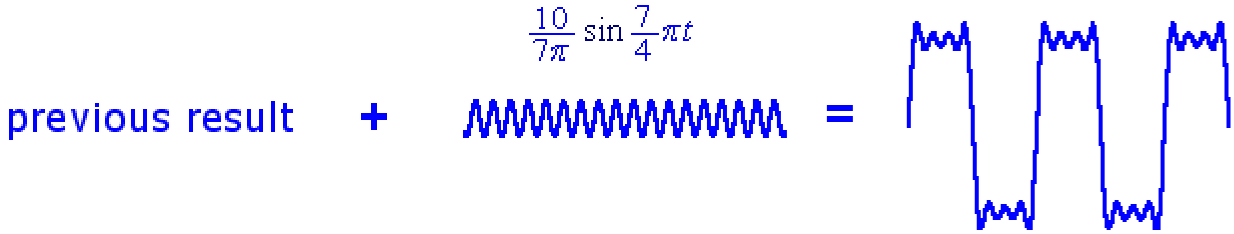
$$f(t) = 2.5 + \frac{10}{\pi}\left(\sin\frac{\pi t}{4} + \frac{1}{3}\sin\frac{3\pi t}{4} + \frac{1}{5}\sin\frac{5\pi t}{4} + \frac{1}{7}\sin\frac{7\pi t}{4} + \ldots\right)$$

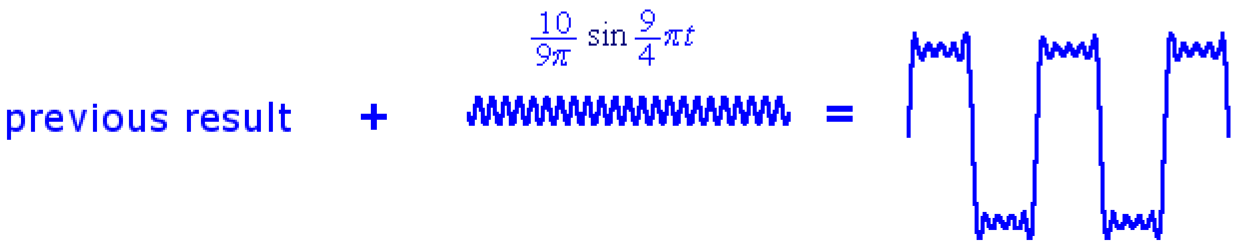
[**NOTE:** Whichever method we choose, *n* must take values 1, 2, 3, ... when we are writing out the series using sigma notation.]

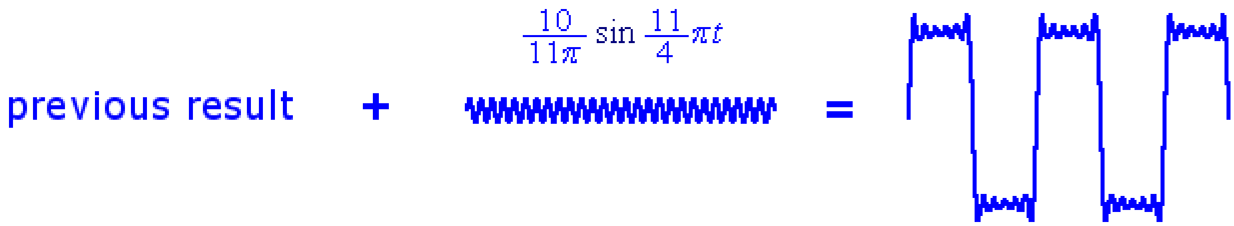
What have we done?

We are adding a series of sine terms (with decreasing amplitudes and decreasing periods) together. The combined signal, as we take more and more terms, starts to look like our original square wave:



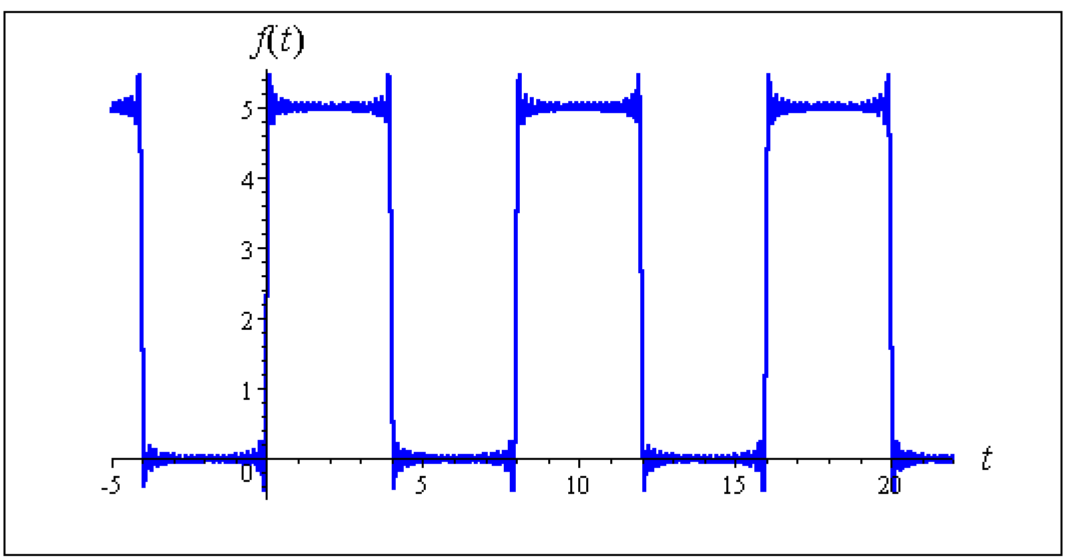






If we graph many terms, we see that our series is producing the required function. We graph the first 20 terms:

$$2.5 + \frac{10}{\pi}\sum_{n=1}^{20} \frac{1}{(2n-1)} \sin \frac{(2n-1)\pi t}{4}$$



Apart from helping us understand what we are doing, a graph can help us check our calculations...

The following video illustrates what we are doing. The equation is not exactly the same, but the concept is. The tone heard at the end is (close to) a "pure" square wave.

Common Case: Period = 2*L*= 2π

If a function is defined in the range -π to π (i.e. period $2L = 2\pi$ radians), the range of integration is 2π and half the range is $L = \pi$.

The Fourier coefficients of the Fourier series $f(t)$ in this case become:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$

and the formula for the Fourier Series becomes:

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

where $n = 1, 2, 3, \ldots$

Example

a) Sketch the waveform of the periodic function defined as:

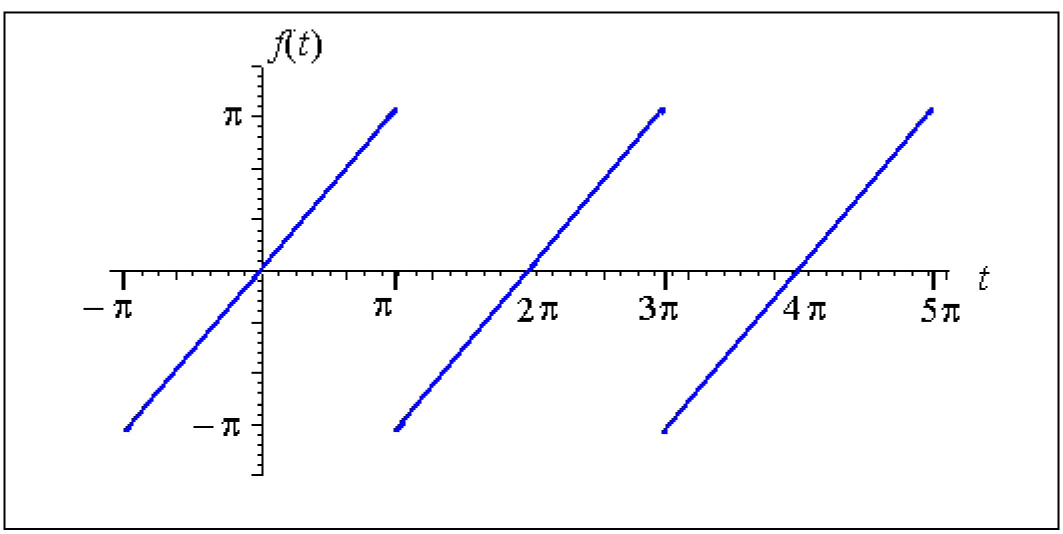
$f(t) = t$ for $-\pi < t < \pi$

$f(t) = f(t + 2\pi)$ for all t.

b) Obtain the Fourier series of $f(t)$ and write the first 4 terms of the series.

Answer

a) Sketch:



b) First, we need to find the **Fourier coefficients** a_0, a_n, and a_b.

$$
\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} t\,dt \\
&= \frac{1}{\pi}\left[\frac{t^2}{2}\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \frac{\pi^2}{2}\right] \\
&= 0
\end{aligned}
$$

Now, using a result from before:

$$\int t\cos nt\,dt = \frac{1}{n^2}(\cos nt + nt\sin nt)$$

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} t\ \cos nt\,dt \\
&= \frac{1}{\pi}\left[\frac{1}{n^2}(\cos nt + nt\sin nt)\right]_{-\pi}^{\pi} \\
&= \frac{1}{\pi n^2}[(\cos n\pi + 0) - (\cos(-n\pi) + 0)] \\
&= \frac{1}{\pi n^2}(\cos n\pi - \cos n\pi) \\
&= 0
\end{aligned}
$$

Once again, using a result from before:

$$\int t\sin nt dt = \frac{1}{n^2}(\sin nt - nt\cos nt)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt$$
$$= \frac{1}{\pi}\int_{-\pi}^{\pi} t\ \sin nt\,dt$$
$$= \frac{1}{\pi}\left[\frac{1}{n^2}(\sin nt - nt\cos nt)\right]_{-\pi}^{\pi}$$
$$= \frac{1}{n^2\pi}([0 - n\pi\cos n\pi] - [0 + n\pi\cos(-n\pi)])$$
$$= \frac{n\pi}{n^2\pi}(-2\cos n\pi)$$
$$= -\frac{2}{n}\cos n\pi$$
$$= -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1}$$

Now for the Fourier Series:

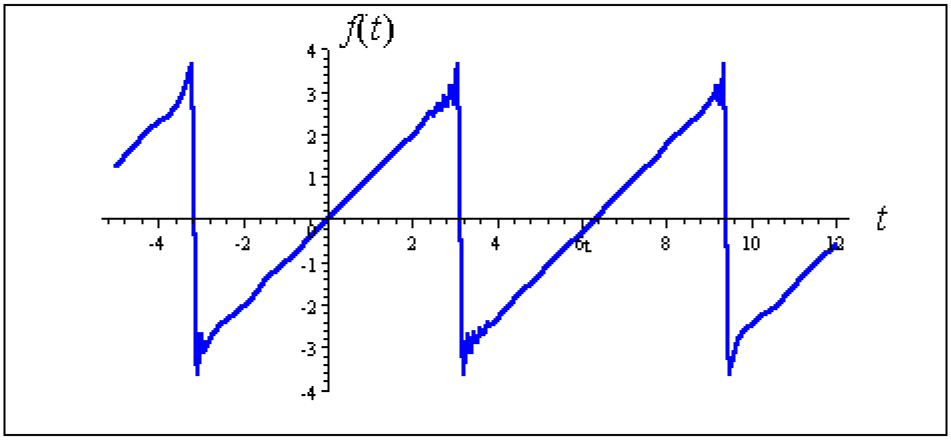
$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nt + b_n\sin nt)$$
$$= \frac{0}{2} + \sum_{n=1}^{\infty}\left((0)\cos nt + \frac{2}{n}(-1)^{n+1}\sin nt\right)$$
$$= \sum_{n=1}^{\infty}\left(\frac{2}{n}(-1)^{n+1}\sin nt\right)$$
$$= 2\sin t - \sin 2t + \frac{2}{3}\sin 3t - \frac{1}{2}\sin 4t + \ldots$$

What have we found?

Let's see an animation of this example using LiveMath.

The graph of the first 40 terms is:

$$\sum_{n=1}^{40}\left(\frac{2}{n}(-1)^{n+1}\sin nt\right)$$



We can express the Fourier Series in different ways for convenience, depending on the situation.

Fourier Series Expanded In Time *t* with period *T*

Let the function $f(t)$ be periodic with period $T = 2L$ where

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2L} = \frac{\pi}{L}.$$

In this case, our lower limit of integration is 0.

Hence the Fourier series is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\omega t + \sum_{n=1}^{\infty} b_n \sin n\omega t$$

where

$$a_0 = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)dt$$

$$a_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\cos n\omega t\, dt \quad b_n = \frac{\omega}{\pi}\int_0^{2\pi/\omega} f(t)\sin n\omega t\, dt$$

(Note: half the range of integration = π/ω)

Fourier Series Expanded in Angular Displacement ω

(**Note:** ω is measured in radians here)

Let the function $f(\omega)$ be periodic with period $2L$.

We let $\theta = \omega t$. This function can be represented as

$$f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi\theta}{L} + b_n \sin\frac{n\pi\theta}{L}\right)$$

where

$$a_0 = \frac{1}{L}\int_0^{2L} f(\theta)d\theta$$

$$a_n = \frac{1}{L}\int_0^{2L} f(\theta)\cos\frac{n\pi\theta}{L}d\theta \quad b_n = \frac{1}{L}\int_0^{2L} f(\theta)\sin\frac{n\pi\theta}{L}d\theta$$