UNIT – 2
FOURIER TRANSFORM

Syllabus: Integral Transforms: Fourier Transform-Complex Fourier Transform, Fourier Sine and Cosine Transforms, Applications of Fourier Transform in Solving the Ordinary Differential Equation.

Any general periodic signal has the automatic property $f(t) = f(\frac{2\pi t}{T})$ where T is the period of the signal. The 2π is "snuck" in because we know that trigonometric functions are good examples of repetition. The complexity of $f(t)$ is irrelevant as long as it repeats itself faithfully. Please keep in mind that 't' for radioastronomy is usually time, but in fact it is an arbitrary variable and so what follows below is applicable provided the variable has functional repetition in some way with a repeat T. Thus spatial repetition is another important variable to which we may apply the theory.

Fourier discovered that such a complex signal could be decomposed into an infinite series made up of cosine and sine terms and a whole bunch of coefficients which can (surprisingly) be readily determined.

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(\frac{2\pi nt}{T}) + \sum_{n=1}^{\infty} b_n \sin(\frac{2\pi nt}{T})$$

If you like, we have decomposed the original function $f(t)$ into a series of basis states. For those of you who like to be creative this immediately begs the question of: is this the only decomposition possible? The answer is no.

The coefficients are "readily" determined by integration.

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\frac{2\pi nt}{T})dt$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(\frac{2\pi nt}{T})dt$$

Introducing complex notation we can simplify all of the above to what you often see in textbooks.

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \exp(-i\frac{2\pi nt}{T})$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\exp(i\frac{2\pi nt}{T})dt$$

Here $c_0 = \frac{1}{2}a_0$, $c_n = \frac{1}{2}(a_n + ib_n)$, and $c_{-n} = \frac{1}{2}(a_n - ib_n)$.

The graphical example below indicates how addition of cosine time function terms are Fourier transformed into coefficients. In this case only $c_n = a_n/2$. Take care the centre line with the big arrow is to mark the axis only – it is **not** part of the coefficient display. Notice also that **two** coefficient lines appear for every frequency. The latter is related to the Nyquist sampling theorem (see below) and is also why the coefficient magnitudes are halved. Notice also the **spacing** of the coefficients to be an integral multiple of $f_0 = 1/T$ with the sign consistent with the input waveform.

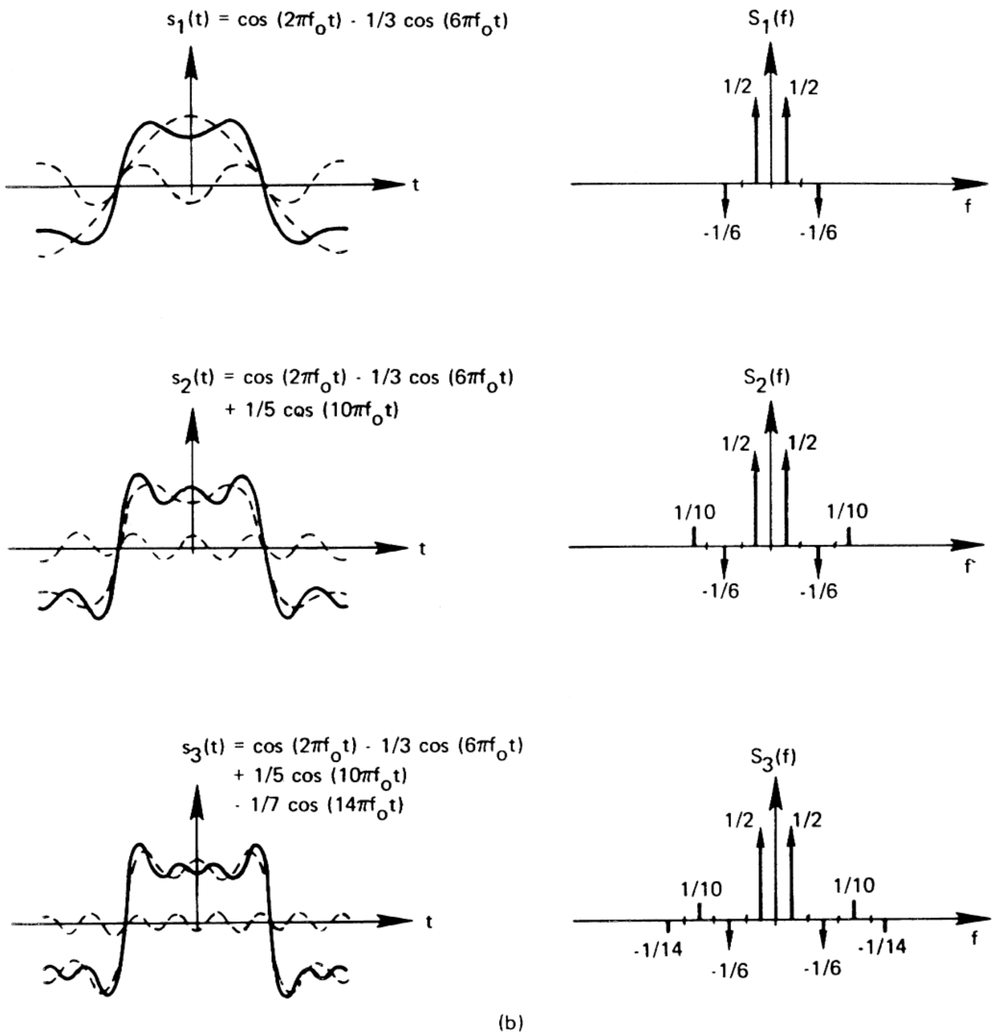

Fig. 2.1(after 'The Fast Fourier Transform', E.O. Brigham, Prentice Hall, 1974)

Exercise:

Q1. Find the Fourier transform of $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$

Soln. $F\{f(x)\} = \int_{-a}^{a} x e^{isx} dx$ =

$\frac{2a}{is}\left(\frac{e^{isx} + e^{-isx}}{2}\right) - \frac{1}{(is)^2}\left(e^{ias} - e^{-ias}\right)$

$F\{f(x)\} = \frac{2i}{s^2}(sinas - ascosas)$

Q2. Find the Fourier sine transform of f(x) = $\begin{cases} sinx, & 0 < x < a \\ 0, & x > a \end{cases}$

Soln. $f_s(s) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) sinsx \, dx$

$$=\sqrt{\frac{2}{\pi}}\int_0^a sinxsinsx\ dx\ +0$$

$$=\frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^a(\cos(s-1)x-\cos(s+1)x)dx$$

$$=-\left[\frac{\sin(s-1)a}{s-1}-\frac{\sin(s+1)a}{s+1}\right]$$

Thinking in terms of the Fourier Transform

- Digital filtering on a static input time sample can be done taking a FFT of the vector and then applying the desired filter shape to the resultant coefficients. Now apply an inverse FFT and the result is a filtered time set. Clean up your old records this way by converting the sound to digital sample files and process them on a PC.

- Continuous digital filtering on a continuous time sample can be thought of as deliberately convolving the incoming signal with a function which is the inverse FT of the filter shape. This is how many digital filters called FIR filters work. They do the job on the fly.

There are a number of techniques to do FFT's quickly and so get power spectra. Correlators use the following simple idea:

1. The autocorrelation of a signal $f(t)$ with a time shifted version of itself $f(t+\tau)$ is given by $A(\tau)=\int f(t)f(t+\tau)dt$.

2. If we take the FT of this in τ space we get, after a fiddle with variable substitution, $\int A(\tau)\exp(i2\pi s\tau)d\tau = F(s)F^*(s)$, provided we assume $f(t)$ is a real function (which of course it will be in our case). But this is exactly the desired power spectrum since $P(s)=F(s)F^*(s)$.

3. Producing a fast autocorrelator using a shift register and a bit of electronics allows $A(\tau)$ to be produced efficiently by continually multiplying a sampled signal with previous samples of itself. The final vector can then be converted into a power spectrum with a single FFT.

The Fourier "Family(Another way to study Fourier)

Fourier Series

If we have a reasonably well behaved, *continuous, periodic* function x(t), then we can approximate x(t) as the weighted sum of simple sinusoids e.g. $x(t) \approx a_o/2 + \sum_n (a_n \cos(2\pi ft) + b_n \sin(2\pi ft))$

if we choose the "weights" a_n and b_n properly. A simple "least squares" procedure yields the well known formulae for calculating a_n and b_n, i.e.

$$a_n = (2/T) \int_{-T/2}^{T/2} x(t) \cos(2\pi ft)\, dt$$

$$b_n = (2/T) \int_{-T/2}^{T/2} x(t) \sin(2\pi ft)\, dt$$

where $f = n/T$

We can use Euler's relation to rewrite the sinusoidal version of Fourier series in terms of exponentials, e.g.

$$x(t) = \sum_n \alpha_n e^{i2\pi ft}$$

$$\text{where } \alpha_n = (1/T) \int_{-T/2}^{T/2} x(t)\, e^{-i2\pi ft}\, dt$$

In the limit of the period T approach infinity, we get the a fit to a non-periodic function x(t) that is the

Fourier Transform

$$\mathbf{X}(f) = \int_{-\infty}^{\infty} x(t)\, e^{-i2\pi ft}\, dt$$

which converts a function of time into a function of frequency
(or a function of space into a function of wavenumber)

x(t) is then represented by **X**(f)

The new function has the same information as x(t) but from a different perspective. If this is done properly, then one can always recover the orginal function x(t) by an **inverse transform**, i.e.

$$x(t) = \int_{-\infty}^{\infty} \mathbf{X}(f)\, e^{+i2\pi ft}\, dt$$

The discrete (digital) form of the transform **(DFT)** for a time series of length N is:

$$X(j\Delta f) = X_j = \sum_{k=0}^{N-1} x_k\, e^{-i2\pi kj/N}$$

and

$$x(k\Delta t) = x_k = 1/N \sum_{j=0}^{N-1} X_j\, e^{i2\pi kj/N}$$

where $\Delta f = 1/N\, \Delta t$

Note the relationship with simply sinuisoids, i.e.

$$e^{-i2\pi kj/N} = \cos(2\pi kj/N) - i \sin(2\pi kj/N)$$

Also note that these equations allow one to compute either X or x for values of j and i respectively that are outside the range of 0 and N-1. These values correspond to the periodic replicas of X and x that are implicit in the discreter formulation.

Now, let's make a simple substitution, letting

$$z = e^{-i2\pi j/N}$$

then the equation for the DFT becomes

$$X(j\Delta t) = X_j = \sum_{k=0}^{N-1} x_k\, e^{-i2\pi kj/N} = \sum_{k=0}^{N-1} x_k\, z^k = X(z).$$

This latter relation is called the

Z transform.

Note that computing the Z transform for a time series is rather trivial:

e.g. if $x_k = 1, 4, 0, -8, 3$

then

$$X(z) = 1z^0 + 4z^1 + 0z^2 - 8z^3 + 3z^4$$

or $$X(z) = 1z + 4z^1 - 8z^3 + 3z^4$$

This real utility of the z transform becomes apparent when implementing convolution via the convolution theorem, i.e.

given x_t as the input to filter f_t, then the output is:

$$s_t = x_t * f_t$$

Using the convolution theorem, this is equivalent to:

$S(z) = X(z) \cdot F(z)$, where these are the z transforms of the their respective time series.

ex. if $x_t = 1, 3$ and $f_t = 4, 2$ then

$S(z) = (1+3z)\cdot(4+2z) = 4 + 14z + 6z^2$ (i.e. polynomial multiplication)

or $s_t = 4, 14, 6$

Deconvolution with the z transform

Z transforms are also very handing for computing inverse filters (deconvolution)

Again, if $s_t = x_t * f_t$ and $S(z) = X(z) \cdot F(z)$.

If we wanter to computer x_t from s_t and f_t, then

$$X(z) = S(z)/F(z) \quad \text{or } X(z) = S(z) \cdot F^{-1}(z)$$

Where $F^{-1}(z) = 1/F(z)$ is the inverse filter for F(z).

We can computer $F^{-1}(z)$ directly by polynomial division:

Example: if $f_t = 2, 1$ and $F(z) = 2+z$,

then $F^{-1}(z) =$

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          .5 -.25z+. 125z^2-.0625z^3+ ..... etc.
2+z |  1
       1+.5z
         -.5z
         -.5z-.25z^2
               .25z^2
```

This form of $F^{-1}(z)$ is called an **Infinite Impulse Response Filter** or **IIR.**

Note that since the higher order terms are decreasing in amplitude, this infinite series is convergent. Thus it is **stable.** This is the case