Engineering Mathematics III
BE 3001

Unit IV

Syllabus: Discrete and Continuous Random Variables, Probability Function , Distribution Function, Density Function , Probability distribution , Mean & Variance of Random Variable.

Random Variable:

A real valued function defined on a sample space is called a Random Variable or a Discrete Random Variable.

A Random Variable assumes only a set of real values & the values which variable takes depends on the chance.

For Example:

a) X takes only a set of discrete values 1,2,3,4,5,6.
b) The values which x takes depends on the chance.

The set values 1,2,3,4,5,6 with their probabilities 1/6 is called the **Probability Distribution** of the variate x.

Continuous Random Variable:
When we deal with variates like weights and temperature then we know that these variates can take an infinite number of values in a given interval. Such type of variates are known as **Continuous Random Variable.**
OR
A Variable which is not discrete i.e. which can take infinite number of values in a given interval $a \le x \le b$, is called **Continuous Random Variable**

Example: Sinx between (0,π) , x is a **Continuous Random Variable.**

Probability Mass Function:
Suppose that *X*: *S* → *A* is a discrete random variable defined on a sample space *S*. Then the probability mass function *p(x)*: *A* → [0, 1] for *X* is defined as:

a) $P(x_i) \ge 0$,for every i=1,2,3..
b) $\sum_{i=1}^{\infty} p(x_i) = 1$
The sum of probabilities over all possible values of a discrete random variables must be equal to 1.

Thinking of probability as mass helps to avoid mistakes since the physical mass is conserved as is the total probability for all hypothetical outcomes *x*.

- The following exponentially declining distribution is an example of a distribution with an infinite number of possible outcomes—all the positive integers:

  $$p(x_i) = \frac{1}{2^i}, i = 1,2,3, \ldots$$

  Despite the infinite number of possible outcomes, the total probability mass is 1/2 + 1/4 + 1/8 + ... = 1, satisfying the unit total probability requirement for a probability distribution.

Probability Density Function:

Let X be a continuous random variable and let the probability of X falling in the infinite interval $(x - \frac{1}{2}dx, x + \frac{1}{2}dx)$ be expressed by f(x)dx, i.e.

$P(x - \frac{1}{2}dx, x + \frac{1}{2}dx)$= f(x)dx

Where f(x) is a continuous function of X & satisfies the following condition:

a) $f(x) \geq 0$
b) $\int_a^b f(x)dx = 1 \;\; if \; a \leq x \leq b$

Then the function is called probability density function of the continuous random variable X.

Continuous Probability Distribution:

The Probability distribution of continuous random variate is called the continuous probability distribution and it is expressed in terms of probability density function.

Cumulative Distribution Function:

The probability that the value of a random variate X is 'x or less than x' is called the Cumulative distribution function of X and is usually denoted by F(x). and it is given by

F(x)=P(X≤x)=$\sum_{x \leq x_i} p(x_i)$

The cumulative distribution functionof a continuous random variable is given by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$$

Some properties of Cumulative Distribution Function:

a) $F(-\infty) = 0$
b) F(X) is non-decreasing function
c) For a distribution variate

$$P(a < x < b) = F(b) - F(a)$$

d) $F(+\infty) = 1$

e) F(x) is a discontinuous function for a discontinuous variate and F(x) is continuous function foe a continuous variate.

Examples:

1) Let *X* be a random variable with PDF given by

$$f(x) = \begin{cases} cx^2 & |x| \leq 1 \\ 0 & otherwise \end{cases}$$

a. Find the constant c

. b. Find *EX* and Var(*X*)

. c. Find *P*(*X*≥12).

Solution: **To find *c*, we can use** $\int_{-\infty}^{\infty} f(x)dx$=1:

$$1=\int_{-1}^{1} cx^2 dx$$

$$1=\frac{2}{3}c$$

Therfore C=$\frac{3}{2}$

To find *EX*, we can write$\int_{-1}^{1} xf(x)dx$=o

In fact, we could have guessed *EX*=0 because the PDF is symmetric around *x*=0. To find Var(*X*), we have

Var(*X*)

=*EX*2−(*EX*)2=*EX*2

=$\int_{-1}^{1} xf(x)dx$

=3/5

To Find *P*(*X*≥12):

P(X≥12)= $\frac{3}{2}\int_{\frac{1}{2}}^{1} x^2 dx$

=7/16.

(2) The probability distribution function for a discrete random variable X is

$$f(x) = \begin{cases} 2k, & x = 1 \\ 3k, & x = 3 \\ 4k, & x = 5 \\ 0, & \text{otherwise} \end{cases}$$

where k is some constant. Please find

(a) k. (b) $P(X > 2)$ (c) $E(X)$ and $Var(X)$

[solution:]

(a)

$$\sum_x f(x) = f(1) + f(3) + f(5) = 2k + 3k + 4k = 9k = 1$$

$$\Leftrightarrow \quad k = \frac{1}{9}.$$

(b)

$$P(X > 2) = P(X = 3 \text{ or } X = 5) = P(X = 3) + P(X = 5)$$

$$= f(3) + f(5) = 3k + 4k = 7k = 7 \cdot \frac{1}{9} = \frac{7}{9} \qquad .$$

(c)

$$u = E(X) = \sum_x xf(x) = 1 \cdot f(1) + 3 \cdot f(3) + 5 \cdot f(5)$$

$$= 1 \cdot \frac{2}{9} + 3 \cdot \frac{3}{9} + 5 \cdot \frac{4}{9} = \frac{31}{9}$$

and

$$Var(X) = \sum_{x} (x-u)^2 f(x)$$

$$= \left(1-\frac{31}{9}\right)^2 \cdot f(1) + \left(3-\frac{31}{9}\right)^2 \cdot f(3) + \left(5-\frac{31}{9}\right)^2 \cdot f(5)$$

$$= \frac{(-22)^2}{81} \cdot \frac{2}{9} + \frac{16}{81} \cdot \frac{3}{9} + \frac{14^2}{81} \cdot \frac{4}{9} = \frac{200}{81}$$

(3) The probability density function for a continuous random variable X is

$$f(x) = a + bx^2, \ 0 \le x \le 1$$
$$0, \ \text{otherwise.}$$

where a, b are some constants. Please find

(a) a, b if $E(X) = \frac{3}{5}$ (b) $Var(X)$.

[solution:]

(a)

$$\int_0^1 f(x)dx = 1 \Leftrightarrow \int_0^1 \left(a + bx^2\right)dx = 1 \Leftrightarrow ax + \frac{b}{3}x^3 \Big|_0^1 = 1$$

$$\Leftrightarrow a + \frac{b}{3} = 1$$

and

$$E(X) = \int_0^1 xf(x)dx = \int_0^1 x\left(a + bx^2\right)dx = \frac{a}{2}x^2 + \frac{b}{4}x^4 \Big|_0^1 = \frac{a}{2} + \frac{b}{4} = \frac{3}{5}$$

Solve for the two equations, we have

$$a = \frac{3}{5},\ b = \frac{6}{5}.$$

(b)

$$f(x) = \frac{3}{5} + \frac{6}{5}x^2,\ 0 \le x \le 1$$
$$0,\ \text{otherwise.}$$

Thus,

$$Var(X) = E[X - E(X)]^2 = E(X^2) - [E(X)]^2 = E(X^2) - \left(\frac{3}{5}\right)^2$$

$$= \int_0^1 x^2 f(x)dx - \frac{9}{25} = \int_0^1 x^2\left(\frac{3}{5} + \frac{6}{5}x^2\right)dx - \frac{9}{25}$$

$$= \frac{1}{5}x^3 + \frac{6}{25}x^5 \Big|_0^1 - \frac{9}{25} = \frac{1}{5} + \frac{6}{25} - \frac{9}{25} = \frac{2}{25}$$

4 ***X:*** the random variable representing the point of throwing a fair dice. Then,

$$P(X = i) = f_x(i) = \frac{1}{6},\ i = 1, 2, 3, 4, 5, 6.$$

Intuitively, the average point of throwing a fair dice is

$$\frac{1+2+3+4+5+6}{6} = 3.5.$$

The expected value of the random variable ***X*** is just the average,

$$E(X) = \sum_{i=1}^{6} i f_x(i) = 1\cdot\frac{1}{6} + 2\cdot\frac{1}{6} + 3\cdot\frac{1}{6} + 4\cdot\frac{1}{6} + 5\cdot\frac{1}{6} + 6\cdot\frac{1}{6} = 3.5 = \text{average point}.$$

(5)The probability density function for a continuous random variable X is

$$f(x) = \begin{cases} \dfrac{x+2}{18}, -2 < x < 4 \\ 0, \text{otherwise} \end{cases}$$

Please find (a) $P(|X|<1)$ **(b)** $P(X^2<9)$ **(c)** $E(X)$ **and** $Var(X)$

[solution:]

(a)

$$P(|X|<1) = P(-1<X<1) = \int_{-1}^{1} \frac{x+2}{18}dx = \left[\frac{x^2}{36}+\frac{x}{9}\right]_{-1}^{1} = \left(\frac{1}{36}+\frac{1}{9}\right) - \left(\frac{1}{36}-\frac{1}{9}\right) = \frac{2}{9}$$

(b) $$P(X^2<9) = P(-3<X<3) = \int_{-3}^{3} \frac{x+2}{18}dx = \int_{-3}^{-2} 0dx + \int_{-2}^{3} \frac{x+2}{18}dx = \left[\frac{x^2}{36}+\frac{x}{9}\right]_{-2}^{3} = \frac{25}{36}$$

(c)

$$E(X) = \mu = \int_{-2}^{4} x \cdot \frac{(x+2)}{18}dx = \int_{-2}^{4}\left(\frac{x^2}{18}+\frac{x}{9}\right)dx = \left[\frac{x^3}{54}+\frac{x^2}{18}\right]_{-2}^{4} = 2.$$

Since

$$E(X^2) = \int_{-2}^{4} x^2 \cdot \frac{(x+2)}{18}dx = \int_{-2}^{4}\left(\frac{x^3}{18}+\frac{x^2}{9}\right)dx = \left[\frac{x^4}{72}+\frac{x^3}{27}\right]_{-2}^{4} = 6,$$

$$Var(X) = E(X-\mu)^2 = E(X^2) - \mu^2 = 6 - 2^2 = 2.$$

(6) The probability distribution functions (discrete random variable) or probability density functions (continuous random variable) for a random variable X are

(a)

$$f(x)=\begin{cases} c\exp(-6x),\ x>0 \\ -cx,\ -1<x\le 0 \\ 0,\ \text{otherwise} \end{cases}$$

(b)

$$f(x)=cx^2\exp(-x^3),\ x>0$$

(c)

$$f(x)=c\left(\frac{1}{3}\right)^x,\ x=0,1,2,\ldots$$

Find c.

[solution:]

(a)

$$\int_{-1}^{0}-cx\,dx+\int_{0}^{\infty}c\exp(-6x)dx=1\Leftrightarrow\left[\frac{-cx^2}{2}\right]_{-1}^{0}+\left[\frac{-c\exp(-6x)}{6}\right]_{0}^{\infty}=1$$
$$\Leftrightarrow\frac{c}{2}+\frac{c}{6}=1\Leftrightarrow\frac{3c+c}{6}=1\Leftrightarrow c=\frac{3}{2}$$

(b)

$$\int_{0}^{\infty}cx^2\exp(-x^3)dx=1\Leftrightarrow\ c\int_{0}^{\infty}\frac{1}{3}\exp(-x^3)dx^3\ =1\Leftrightarrow\left[\frac{-c\exp(-x^3)}{3}\right]_{0}^{\infty}=1$$
$$\Leftrightarrow\frac{c}{3}=1\Leftrightarrow c=3$$

(c)

$$\sum_{x=0}^{\infty}c\left(\frac{1}{3}\right)^x=1\Leftrightarrow\ c\left[1+\frac{1}{3}+\left(\frac{1}{3}\right)^2+\cdots\right]=1\Leftrightarrow c\left(\frac{1}{1-\frac{1}{3}}\right)=1\Leftrightarrow\frac{3c}{2}=1$$
$$\Leftrightarrow c=\frac{2}{3}$$

Example: If $f(x)=cx^2, 0<x<1$. Find the value of c and determine the probability that $\frac{1}{3} < x < \frac{1}{2}$

Solution: By property of p.d.f. we have, $\int_0^1 f(x)dx = 1$

So $\int_0^1 cx^2\, dx = 1$, or $c[\frac{x^3}{3}]_0^1 = 1$, so $c = 3$

Consequently $f(x) = 3x^2 : \quad 0 < x < 1$

Again $P(\frac{1}{3} < X < \frac{1}{2}) = \int_{\frac{1}{3}}^{\frac{1}{2}} 3x^2 \; dx = \frac{1}{8} - \frac{1}{27} = \frac{19}{216}$

Example: For the distribution $dF = sinx\, dx, 0 \le x \le \pi/2.$ Find Mode and Median.

Solution: Here $f(x) = sinx$, $0 \le x \le \frac{\pi}{2}$

(a) For Mode: $f'(x) = 0$ & $f''(x) < 0$, $f'(x) = 0 \Rightarrow cosx = 0 \Rightarrow x = \frac{\pi}{2}$ &

$[f''(x)]_{x=\frac{\pi}{2}} = -1 < 0$, Hence mode $= \frac{\pi}{2}$

Let M_d be median, then $\int_0^{M_d} sinx\, dx = \frac{1}{2} \Rightarrow M_d = \pi/3$

(b) Mean $= \mu_1' = \int_0^{\frac{\pi}{2}} (x-0)f(x)dx = \int_0^{\frac{\pi}{2}} xsinx\, dx = 1$ &

Variance $= \mu_2 = \int_0^{\frac{\pi}{2}} (x-1)^2 sinx\, dx = \pi - 3$

Continuous random variable – infinite number of values with no gaps between the values. [You might consider drawing a line, the sweeping hand on a clock, or the analog speedometer on a car.]

In this section, we restrict our discussion to discrete probability distributions. Each probability distribution must satisfy the following two conditions.

1. $\sum P(x) = 1$ where x assumes all possible values of the random variable

2. $0 \leq P(x) \leq 1$ for every value of x

As we found the mean and standard deviation with data in descriptive statistics, we can find the mean and standard deviation for probability distributions by using the following formulas.

1. $\mu = \sum[x \cdot P(x)]$ **mean** of probability distribution
2. $\sigma^2 = \sum[(x-\mu)^2 \cdot P(x)]$ **variance** of probability distribution
3. $\sigma^2 = \sum[x^2 \cdot P(x)] - \mu^2$ **variance** of probability distribution
4. $\sigma = \sqrt{\left[\sum[x^2 \cdot P(x)] - \mu^2\right]}$ **standard deviation** of probability distribution