Unit-1

Elasto-dynamics

Syllabus:

Simple Harmonic Motion, Electric Flux, displacement vector, Columb law, Gradient, Divergence, Curl, Gauss Theorem, Stokes theorem, Gauss law in dielectrics, Maxwell’s equation: Integral & Differential form in free space, isotropic dielectric medium.

Periodic motion:

If an object repeats its motion on a definite path after a regular time interval then such type of motion is called periodic motion.

1) Vibratory motion or oscillatory motion
2) Uniform circular motion
3) Simple harmonic motion

Vibratory motion:

If a body in periodic motion moves to and fro about a definite point on a single path, the motion of the body is said to be vibratory or oscillatory motion.

<u>Mean or equilibrium position:</u>

The point on either side of which the body vibrates is called the mean position or equilibrium position of the motion.

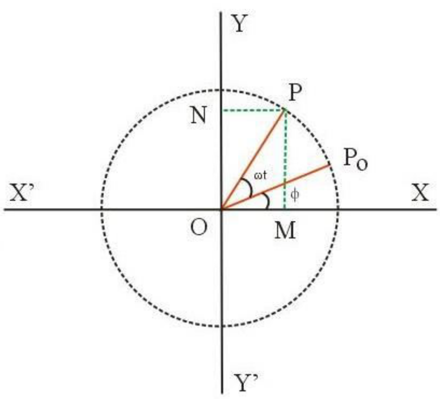
<u>Time period:</u>

The definite time after which the object repeats its motion, is called time period and it is denoted by T.

<u>Frequency:</u>

The number of complete oscillation in one second is called the frequency of that body, it is represented by the letter f or n or ϑ its unit is Hz.

Uniform circular motion:



Figure(1): Uniform circular motion

Let an object is moving on a circular path of radius r with uniform angular velocity $\omega = \frac{2\pi}{T}$.

In right angle triangle ΔOMP

$$\angle POM = \omega t + \phi$$

$$\frac{OM}{OP} = \cos(\omega t + \phi)$$

$$\frac{x}{r} = \cos(\omega t + \phi)$$

$$x = r.\ \cos(\omega t + \phi)$$

But $\omega = \frac{2\pi}{T}$

so $$x = r.\ \cos\left(\frac{2\pi}{T} t + \phi\right)$$

Similarly

$$\frac{MP}{OP} = \sin(\omega t + \phi)$$

$$\frac{y}{r} = \sin(\omega t + \phi)$$

$$y = r.\ \sin(\omega t + \phi)$$

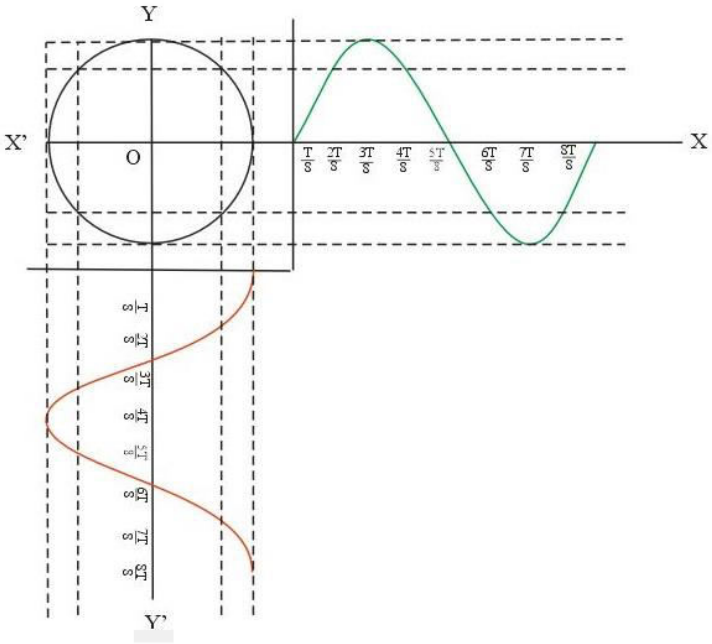
$$y = \mathrm{r}.\ \sin\left(\frac{2\pi}{T} t + \phi\right)$$

Both equation (1) and (2) represents the uniform circular motion.

Simple Harmonic Motion (SHM):

When a body moves periodically on a straight line on either side of a point, the motion is called the simple harmonic motion.

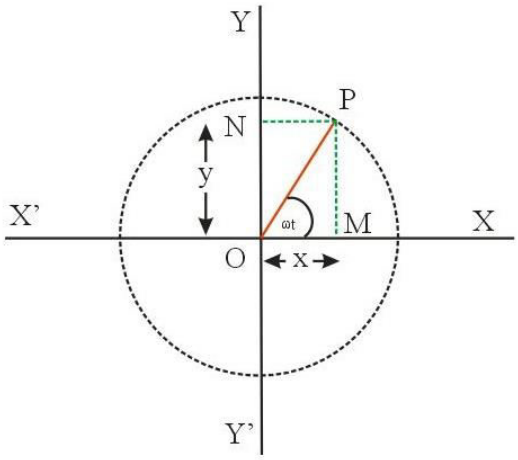
Graphical representation of SHM



Figure(2): Graphical representation of SHM

Displacement in SHM:

Let a particle is moving on a circular path with uniform angular velocity "ω" and the radius of the circular path is "r"; then movement of the point on their axis i.e. N and M is the SHM about the mean position O



Figure(3): SHM

Let at time $t = 0$ the particle is on point A and after time t the position of the particle is P then

In ΔOPN

$$\frac{y}{r} = \sin(\omega t)$$

$$y = r.\ \sin(\omega t) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$y = r.\ \sin\frac{2\pi}{T}(t)$$

This equation represents the displacement of foot dropped from the position of particle on $Y - axis$

Velocity in SHM:

Differentiating equation (1) with respect to t we get-

$$\frac{dy}{dt} = \frac{d}{dt}(r.\ \sin\ \omega t)$$

$$\frac{dy}{dt} = \mathrm{r\omega\ cos}\ \omega t \quad \ldots\ldots\ldots\ldots(2)$$

$$\frac{dy}{dt} = r\omega\sqrt{1 - \sin^2\omega t}$$

$$\frac{dy}{dt} = \omega\sqrt{r^2 - r^2\sin^2\omega t}$$

$$\frac{dy}{dt} = \omega\sqrt{r^2 - y^2} \quad \text{Using (1)}$$

(i) In equilibrium condition $y = 0$

So

$$\frac{dy}{dt} = \omega\sqrt{r^2 - 0^2}$$

$$\frac{dy}{dt} = r\omega$$

(ii) In the position of maximum displacement i.e. $y = r$

So

$$\frac{dy}{dt} = \omega\sqrt{r^2 - r^2}$$

$$\frac{dy}{dt} = 0$$

Acceleration:

Again differentiating equation (2) we get-

$$\frac{d^2y}{dt^2} = \frac{dv}{dt}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}(\mathrm{r\omega\ cos}\ \omega t)$$

$$\frac{d^2y}{dt^2} = -\mathrm{r\omega^2\ sin}\ \omega t$$

$$\frac{d^2y}{dt^2} = -\omega^2 y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

$$\frac{d^2y}{dt^2} + \omega^2 y = 0$$

This is a second order differential equation which denotes the equation of SHM in the differential form

Again by equation (3)

$$\frac{d^2y}{dt^2} = -\omega^2 y$$

Multiplying by m i.e. the mass of the particle executing SHM then

$$m\frac{d^2y}{dt^2} = -m\omega^2 y$$

$$F = -m\omega^2 y$$

Here negative sing shows that the direction of displacement and acceleration are opposite to one another

So $$F \propto -y$$

$$\because \omega^2 = constant$$

Time period and frequency:

$$a = \omega^2 y$$

$\Rightarrow$

$$\omega = \sqrt{\frac{a}{y}}$$

$$\frac{2\pi}{T} = \sqrt{\frac{a}{y}}$$

$$\frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{a}{y}}$$

$$\nu = \frac{1}{2\pi}\sqrt{\frac{a}{y}}$$

And $$T = 2\pi\sqrt{\frac{y}{a}}$$

Question: A uniform circular motion is given by the equation $x = 10 \sin(20t + 0.5)$, find

1) Amplitude
2) Angular frequency
3) Time period
4) Phase

Sol: Given: $x = 10 \sin(20t + 0.5)$

Comparing the given equation with the standard equation of uniform circular motion i.e. $x = A \sin(\omega t + \phi)$

We get

$A = 10\,m$

$\omega = 20\,\frac{rad}{s}$

$\nu = \frac{\omega}{2\pi} = \frac{20}{2\pi} = 0.318\,s$

$T = \frac{2\pi}{T} = \frac{2\pi}{20} = 0.314\,s$

Question: A particle is moving with SHM in a straight line. When the displacement of the particle from equilibrium position has values x_1 and x_2, the corresponding position has valocities v_1 and v_2 show that the time period of oscillation is given by

$$T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$$

Sol: In the SHM the velocity is given by-

$$v = \omega\sqrt{r^2 - x^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

At x_1 velocity is v_1

So

$$v_1 = \omega\sqrt{r^2 - x_1^2}$$

Squaring both sides

$$v_1^2 = \omega^2(r^2 - x_1^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Again at x_2 the velocity is v_2

So

$$v_2^2 = \omega^2(r^2 - x_2^2) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

By equation (2) and (3)

$$v_1^2 - v_2^2 = \omega^2(r^2 - x_1^2) - \omega^2(r^2 - x_2^2)$$

$$v_1^2 - v_2^2 = \omega^2(x_2^2 - x_1^2)$$

$$\frac{(v_1^2 - v_2^2)}{(x_2^2 - x_1^2)} = \omega^2$$

$$\omega^2 = \frac{(v_1^2 - v_2^2)}{(x_2^2 - x_1^2)}$$

$$\omega = \sqrt{\frac{(v_1^2 - v_2^2)}{(x_2^2 - x_1^2)}} \quad \text{..............................} \quad (4)$$

Now $\omega = \frac{2\pi}{T}$

So

$$T = 2\pi\sqrt{\frac{x_2^2 - x_1^2}{v_1^2 - v_2^2}}$$

Question: If the earth were a homogeneous sphere and a straight hole was bored in it through the centre, then a body dropped in the hole, execute SHM. Calculate the time period of its vibration. Radius of the earth is $6.4 \times 10^6 m$ and $g = 9.8\ ms^{-2}$

Solution: The time period of oscillation executed by the body dropped in the hole along the diameter of earth

$$T = 2\pi\sqrt{\frac{R}{g}} = 2\pi\sqrt{\frac{6.4 \times 10^6}{9.8}} = 5077.5\ s$$

Energy of a particle executing SHM:

A particle executing SHM possess potential energy (U) on the account of its position and kinetic energy (KE) on account of motion.

Potential energy:

We know that the acceleration in a simple harmonic motion is directly proportional to the displacement and its direction is towards the mean position

$$a = -\omega^2 y$$

Let m is the mass of particle executing SHM then the force acting on the particle will be-

$$F = m.\ a$$

$$F = -m\omega^2 y$$

If the particle undergoes an infinitesimal displacement against the restoring force, then the small amount of work done against the restoring force is given by

$$dW = (-F).\ dy$$

Here negative sign shows that the restoring force is acting the displacement than

$$dW = m\,\omega^2\,y\,dy$$

So the total amount of work done

$$W = m\omega^2 \int y\,dy$$

$$W = \frac{1}{2} m\omega^2 y^2$$

This work done is equal to the potential energy U of the particle at displacement y

i.e. $$U = \frac{1}{2} m\omega^2 y^2$$

Kinetic energy:

If v is the velocity of the particle executing SHM, when the displacement is y then kinetic energy KE

$$KE = \frac{1}{2} m v^2$$

But for SHM $v = \omega\sqrt{(r^2 - y^2)}$

Where r is the amplitude of SHM

So

$$KE = \frac{1}{2} m \left(\omega\sqrt{r^2 - y^2}\right)^2$$

$\Rightarrow$ $$KE = \frac{1}{2} m\omega^2 (r^2 - y^2) \quad \text{..................................} \quad (2)$$

Total energy:

Now the total energy

$$E = U + KE$$

$\Rightarrow$ $$E = \frac{1}{2} m\omega^2 y^2 + \frac{1}{2} m\omega^2 (r^2 - y^2)$$

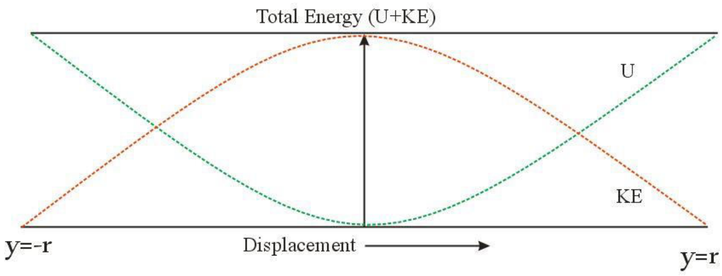
$\Rightarrow$ $$E = \frac{1}{2} m\omega^2 y^2 + \frac{1}{2} m\omega^2 r^2 - \frac{1}{2} m\omega^2 y^2$$

$\Rightarrow$ $$E = \frac{1}{2} m\omega^2 r^2$$

Thus we find that the total energy:

1) $E \propto m$
2) $E \propto$ of SHM
3) $E \propto ^2$ of SHM

Graphical representation of total energy of SHM



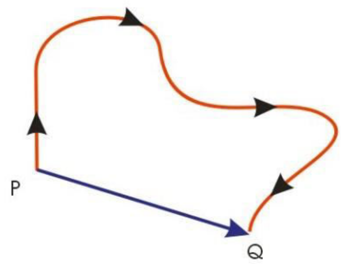
Figure(4): Total energy of SHM

Position vector:

A position vector expresses the position of a point P in space in terms of a displacement from an arbitrary reference point O (typically the origin of a coordinate system). Namely, it indicates both the distance and direction of an imaginary motion along a straight line from the reference position to the actual position of the point.

Displacement Vector:

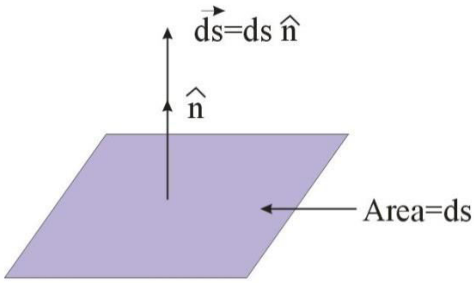
A **displacement** is the shortest distance from the initial to the final position of a point P. Thus, it is the length of an imaginary straight path, typically distinct from the path actually travelled by particle or object. A displacement vector represents the length and direction of this imaginary straight path.



Figure(5): Displacement vector

Area Vector:

In many problems the area is treated as a vector, an area element ds is represented by $\overrightarrow{ds}$, such that the area representing the area vector $\overrightarrow{ds}$ is perpendicular to the area element. The length of the vector $\overrightarrow{ds}$ represents the magnitude of the area element ds

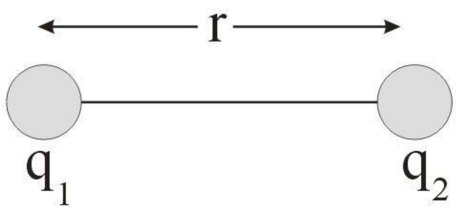


Figure(6): Area vector

Coulomb's Law:

According to it the force of attraction or repulsion between the two point charges is directly proportional to the product of the magnitude of the charges and inversely proportional to the square of the distance between them.

If two charges q_1 and q_2 are separated at a distance r form one another then the force between these charges will be-



Figure(7): Two electric charges separated a distance r

i) Force is proportional to the product of the magnitude of the charges i.e. $f \propto q_1 . q_2$

ii) The force is inversely proportional to the distance between the charges i.e. $f \propto \frac{1}{r^2}$

So

$$f \propto \frac{q_1 . q_2}{r^2}$$

$$f = k\frac{q_1 . q_2}{r^2}$$

Where K is a proportionality called electrostatic force constant, its value depends on the nature of the medium in which the two charges are located and also the system of units adopted to measure q_1, q_2 and r.

So

$$f = k\frac{q_1 . q_2}{r^2}$$

***Case 1:**(when the medium between the charges is air or vacuum)*

As we know that the force between the charges is given as-

$$f = k\frac{q_1.q_2}{r^2}$$

If we put $q_1 = q_2 = 1\,C$ and $r = 1\,m$ then

$$f = k$$

So K is the force feels by two charges of $1\,C$ placed $1\,m$ apart from one another in vacuum or free space.

Its value is $K = 9 \times 10^9\ newton \times meter^2 \times coulomb$

***Case 2:**(When the medium between the charges is other than the vacuum)*

If the changes are located in any other medium then

$$k = \frac{1}{4\pi\varepsilon_0}.\frac{1}{\varepsilon_r} = 9 \times 10^9.\frac{1}{\varepsilon_r}$$

Where ε_r is the dielectric constant of relative permittivity.

Putting this value in equation (1) we get

$$F' = \frac{1}{4\pi\varepsilon_0\varepsilon_r}.\frac{q_1.q_2}{r^2}$$

Where F' is the force in the medium

$$F' = \frac{1}{4\pi\varepsilon}.\frac{q_1.q_2}{r^2}$$

Where $\varepsilon = \varepsilon_r\varepsilon_0$ is called the relative permittivity of the medium.

Vector form of the Coulomb's Law

Consider two like charges q_1 and q_2 present at A and B in vacuum at a distance r apart. The two charges will exert equal repulsive force on each other,

Let $\vec{F}_{12}$ be the force on charge q_1 due to the charge q_2 and $\vec{F}_{21}$ be the force on charge q_1 due to charge q_2.

According to the Coulombs' law, the magnitude of force on charge q_1 and q_2 is given by

$$|\vec{F}_{12}|.|\vec{F}_{21}| = \frac{1}{4\pi\varepsilon_0}\frac{q_1.q_2}{r^2} \qquad \text{..............................} \qquad (1)$$

Let $\hat{r}_{12}$ and $\hat{r}_{21}$ are the unit vectors in the direction from q_1 to q_2 and vice versa.

So the force $\vec{F}_{12}$ is along the direction of unit vector $\hat{r}_{21}$, we have

$$\vec{F}_{12} = \frac{1}{4\pi\varepsilon_0}.\frac{q_1.q_2}{r^2}\hat{r}_{21}$$

And

$$\vec{F}_{21} = \frac{1}{4\pi\varepsilon_0}.\frac{q_1.\ q_2}{r^2}\hat{r}_{12}$$

These two equations show the Coulombs' law in vector form.

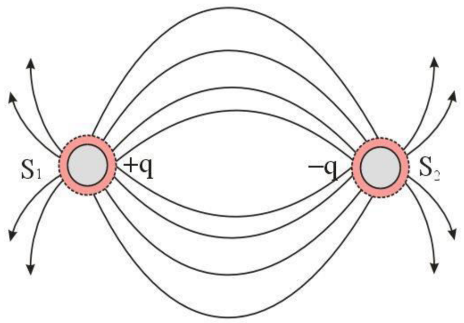
Electric flux:

Number of electric lines of forces passing normally through the surface, when held in the electric field. It is denoted by ϕ_E. There are two types of electric flux-

1. Positive electric flux: When electric lines of forces leave any body through its surface it is considered as positive electric flux.
2. Negative electric flux: When lines of forces enter through any surface, it is considered as the negative electric flux.

Measurement: Let us consider a small area $\overrightarrow{ds}$ of a closed surface S. The electric field $(\vec{E})$ produced due to the charge q will be radially outwards which will be along $\hat{n}$. Now the normal to the surface area ds is $\overrightarrow{ds}$ as shown in the figure, hence the angle between $\overrightarrow{ds}$ and $\hat{n}$ is θ

So the electric lines of forces from the surface area will be given as-



Figure(8): Electric flux

$$d\phi = \vec{E}.\overrightarrow{ds}$$

$$d\phi = (E \cos \theta).ds \text{.......(1)}$$

Where $E \cos \theta$ is the component of electric field $\vec{E}$ along $\overrightarrow{ds}$.

Hence the electric flux through a small elementary surface area is equal to the product of the small area and normal component of $\vec{E}$ along the direction of the elementary area $\overrightarrow{ds}$.

Over the hole surface,

$$\phi_E = \oint E\ ds\cos\theta$$

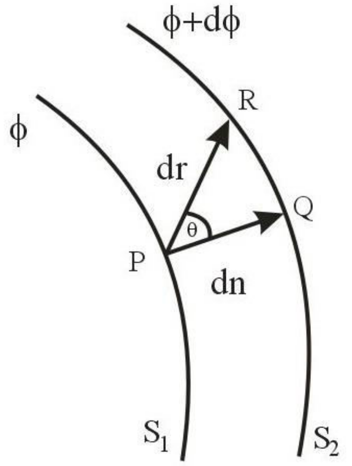
$$\phi_E = \oint \vec{E}.\overrightarrow{ds} \qquad \text{..............................} \qquad (2)$$

Gradient of a scalar field:

The gradient of a scalar function ϕ is a vector whose magnitude is equal to maximum rate of chcnge of scalar function ϕ with respect to the space variable $(\vec{\nabla})$ and has direction along that change.

$$grad\ \phi = \frac{\partial \phi}{\partial n}\hat{n}$$

In the scalar field let there be two level surfaces S_1 and S_2 close together characterised by the scalar function ϕ and $\phi + d\phi$ respectively. Consider point P and R on the level surfaces S_1 and S_2 respectively. Let $\vec{r}$ and $\vec{r} + \overrightarrow{dr}$ be the position vector of P and R. Then $\overrightarrow{PR} = \overrightarrow{dr} = \hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz$



Figure(9): Gradient of a scalar field

Now as ϕ is a function of (x, y, z).

$$\phi = \phi(x, y, z)$$

Then the total differentiation of this function can be given as

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$

$$d\phi = \left(\hat{\imath}\frac{\partial \phi}{\partial x} + \hat{\jmath}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}\right) \cdot (\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz)$$

$$d\phi = (\vec{\nabla}\phi)\overrightarrow{dr} \quad \text{...} \quad (1)$$

Agian if dn represents the distance along the normal from point P to the surface S_2 to point Q, then

$$PQ = dn$$

In the ΔPQR

$$\frac{dn}{dr} = \cos\theta$$

$$dn = dr\cos\theta$$

Now if we consider a unit vector along dn as $\hat{n}$

then

$$dn = \overrightarrow{dr}.\ \hat{n} \quad \text{...} \quad (2)$$

If we proceed form P to Q then value of scalar function ϕ increases by an amount $d\phi$

$\because$

$$d\phi = \frac{\partial \phi}{\partial n}dn$$

$$d\phi = \frac{\partial \phi}{\partial n}(\overrightarrow{dr}.\ \hat{n}) \quad \text{[Using (2).................................}\quad (3)$$

By equation (1) and (2)

$$(\vec{\nabla}.\ \phi)\ \overrightarrow{dr} = \frac{\partial\phi}{\partial n}(\overrightarrow{dr}.\ \ \hat{n})$$

$$(\vec{\nabla}.\ \phi) = \frac{\partial\phi}{\partial n}\hat{n}$$

$$grad\ \phi = \frac{\partial\phi}{\partial n}\hat{n}$$

Note: $\vec{\nabla} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$ is called del or Nabla operator.

Note:

$$grad\ \phi = \vec{\nabla}.\ \phi$$

$$grad\ \phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\ .\ \phi$$

$$grad\ \phi = \left(\hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right)$$

Note: The gradient of a scalar field has great significant in physics. The negative gradient of electric potential of electric field at a point represents the electric field at that point. i.e. $\vec{E} = -grad\ V$

Note: The gradient of a scalar field is a vector quantity.

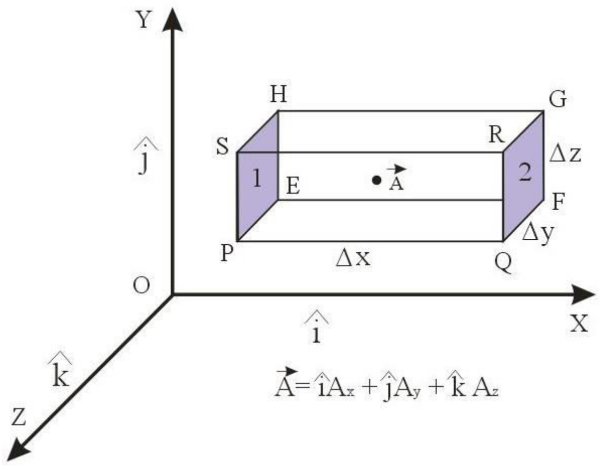
Divergence of a vector field:

The divergence of a vector field $\vec{A}$ at a certain point $P(x, y, z)$ is defined as the outward flux of the vector field $\vec{A}$ per unit volume enclosed through an infinitesimal closed surface surrounding the point "P".

$$div\,\vec{A} = \lim_{\tau \to 0} \frac{\iint_s \vec{A} \cdot \overrightarrow{ds}}{\tau}$$

$$div\,\vec{A} = \lim_{\tau \to 0} \frac{\delta\phi}{\tau}$$

Consider a infinitesimal rectangular box with sides $\Delta x, \Delta y, \Delta z$ and one corner at the point $P(x, y, z)$ in the region of any vector function $\vec{A}$ with rectangular faces perpendicular to co-ordinates axis.



Figure(10): divergence of a vector field

The flux emerging outwards from surface $QFGR$ i. e. surface 2, $\delta\phi_{2x}$ $= \iint_{QFGR} \bar{A}_{2x} \cdot ds$

$$\delta\phi_{2x} = \iint_{QFGR} (\hat{\imath}\bar{A}_{2x} + \hat{\jmath}\bar{A}_{2y} + \hat{k}\bar{A}_{2z}) \cdot (\hat{\imath}\Delta y, \Delta z)$$

Where $\overline{A_2}$ is the average of the vector function over the surface $QFGR$ i.e. surface 2

$$\delta\phi_{2x} = \iint_{QFGR} \overline{A_{2x}} \cdot \Delta y. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Similarly

The flux emerging out from the surface $PEHS$ i.e. surface 1, $\delta\phi_{1x}$ $= \iint_{PEHS} \bar{A}_{1x} \cdot ds$

$$\delta\phi_{1x} = \iint_{PEHS} (\hat{\imath}\bar{A}_{1x} + \hat{\jmath}\bar{A}_{1y} + \hat{k}\bar{A}_{1z}) \cdot (-\hat{\imath}\Delta y, \Delta z)$$

$$\delta\phi_{1x} = \iint_{PEHS} -\bar{A}_{1x} \cdot \Delta y. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Thus net outwards flux of vector $\vec{A}$ through the two faces perpendicular to X axis,

$$\delta\phi_x = \delta\phi_{2x} + \delta\phi_{1x}$$

$$\delta\phi_x = \iint (\bar{A}_{2x} - \bar{A}_{1x})(\Delta y.\ \Delta z) \qquad \text{..................} \qquad (3)$$

But $\quad (\overline{A_{2x}} - \overline{A_{1x}}) = A_x(x + \Delta x, y, z) - A_x(x, y, z)$

$$(\overline{A_{2x}} - \overline{A_{1x}}) = \frac{\partial A_x}{\partial x}\Delta x \qquad \text{..} \qquad (4)$$

Where $\frac{\partial A_x}{\partial x}$ is the variation of A_x with distance along X axis by equation (2) and (3)

Thus net outward flux of vector function $\vec{A}$ through the two faces perpendicular to X axis

$$\delta\phi_x = \frac{\partial A_x}{\partial x}\Delta x\Delta y, \ \Delta z \qquad \text{[Using equation (3)}$$

Similarly perpendicular to Y axis

$$\delta\phi_y = \frac{\partial A_y}{\partial y}\Delta x\Delta y\Delta z$$

Similarly perpendicular to Z axis

$$\delta\phi_z = \frac{\partial A_z}{\partial z}\Delta x\Delta y\Delta z$$

Therefore whole outward flux through infinitesimal box

$$\delta\phi = \delta\phi_x + \delta\phi_y + \delta\phi_z$$

$$\delta\phi = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\Delta x\Delta y\Delta z$$

Now $div\vec{A}$ at any point, which is the flux enclosed per unit infinitesimal volume surrounding that point is given by-

$$div\,\vec{A} = \lim_{\Delta x\Delta y\Delta z\to 0}\frac{\delta\phi}{\Delta x\Delta y\Delta z}$$

$$div\,\vec{A} = \lim_{\Delta x\Delta y\Delta z\to 0}\frac{\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\Delta x\Delta y\Delta z}{\Delta x\Delta y\Delta z}$$

$$div\,\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$div\,\vec{A} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z\right)$$

$$div\,\vec{A} = \vec{\nabla}.\vec{A}$$

Note: Divergence of a vector field is a scalar quantity.

Note: If $div\,\vec{A} = +ve$

it indicates the existence of the source of fluid at that point.

Note: If $div\,\vec{A} = \quad -ve$

It means fluid is flowing towards the point and thus there exist a sink for the fluid.

Note: If $div\,\vec{A} = \quad 0$

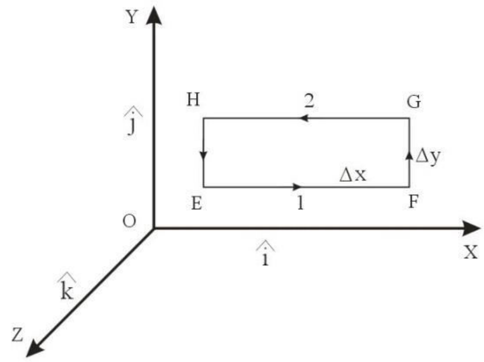
It means the fluid is flowing continuously from that point. In other words this means that the flux of the vector function entering and leaving this region is equal. This condition is called solenoidal vector.

Curl of a vector field:

If $\vec{A}$ is any vector field at any point P and δs an infinitesimal test area at point P then $curl$

$$curl\ \vec{A} = \lim_{\delta s \to 0} \frac{\oint \vec{A}.\ \overrightarrow{dr}}{\delta s}\ \hat{n}$$

Let us consider an infinitesimal rectangular area $EFGH$ with sides Δx and Δy parallel to $X - Y$ plane in the region of vector function $\vec{A}$.



Figure(11): Curl of a vector field

Let the coordinate of E be (x, y, z)

A_x, A_y, A_z are the Cartesian components of A at P then

$$\vec{A} = (\hat{i}A_x + \hat{j}A_y + \hat{k}A_z)$$

Now the line integral of vector field along the path EF (T_{1x})

$$= \int_{EF} \vec{A}\ \overrightarrow{dr}$$

$$= (\hat{i}\bar{A}_{1x} + \hat{j}\bar{A}_{1y} + \hat{k}\bar{A}_{1z}).\ (\hat{\imath}\,\Delta x)$$

$$= \bar{A}_{1x}\,\Delta x$$

Where $\bar{A}_{1x}$ is the average value of X - component of the vector function over the path EF

Similarly for the Path GH

$$T_{2x} = \int_{EF} \vec{A}\ \overrightarrow{dr}$$

$$= (\hat{i}\bar{A}_{2x} + \hat{j}\bar{A}_{2y} + \hat{k}\bar{A}_{2z}).\ (-\hat{\imath}\,\Delta x)$$

$$= -\bar{A}_{2x}\,\Delta x$$

Where $\bar{A}_{2x}$ is the average value of X - component of vector function over the path GH.

Hence the contribution to line integral $\oint \vec{A}.\overrightarrow{dr}$ form two path (EF and GH) parallel to X - axis is

$$T_x = T_{1x} - T_{2x}$$

$$= -(A_{2x} - A_{1x})\Delta x$$

As the rectangle is infinitesimal the difference between the average of A_x ($i.e.$ $\bar{A}_{2x} - \bar{A}_{1x}$) along these two paths may be approximated to the difference between the values of A_x at E and H

Thus-

$$\bar{A}_{2x} - \bar{A}_{1x} = A_{Hx} - A_{Ex}$$

$$\bar{A}_{2x} - \bar{A}_{1x} = A_x(x, y + \Delta y, z) - A_x(x, y, z)$$

$$\bar{A}_{2x} - \bar{A}_{1x} = \frac{\partial A_x}{\partial y}\,\Delta y$$

Hence the contribution to the line integral $\oint_{EFGH} \vec{A}.\ \overrightarrow{dr}$ from the path EF and GH

$$T_x = \frac{\partial A_x}{\partial y}\Delta y\Delta x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Similarly by the path FG and HE

$$T_y = \frac{\partial A_y}{\partial x}\Delta x\Delta y \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Therefore the line integral along the whole rectangular $EFGH$ form (2) and (3) is given by-

$$T = \oint_{EFGH} (T_x + T_y).\ \overrightarrow{dr}$$

$$T = \oint_{EFGH} \vec{A}.\overrightarrow{dr}$$

$$\delta T = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\Delta y\Delta x \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (4)$$

Now $$(curl\ A)_z = \lim_{\Delta y\Delta x\to 0} \frac{\delta T}{\delta s}$$

$$(curl\ A)_z = \lim_{\Delta y\Delta x\to 0} \frac{\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\Delta y\Delta x}{\Delta y\Delta x}$$

$$(curl\ A)_z = \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5)$$

Similarly

$$(curl\ A)_y = \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

and $$(curl\ A)_x = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (7)$$

Summing up the results given in (5), (6) and (7) we get

$$curl\ \vec{A} = \hat{\imath}(curl\ A)_x + \hat{\jmath}(curl\ A)_y + \hat{k}(curl\ A)_z$$

$$curl\ \vec{A} = \hat{\imath}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$curl\ \vec{A} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{bmatrix}$$

$$curl\ \vec{A} = \vec{\nabla} \times \vec{A}$$

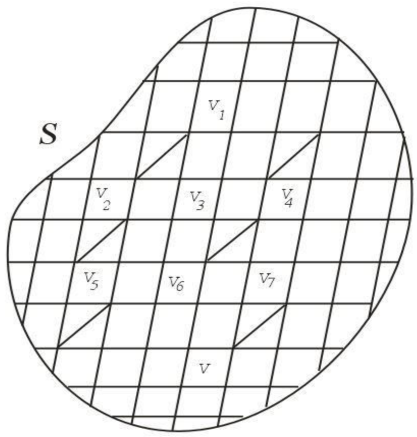
Note: The curl of a vector field is sometime called circulation or rotation or simply rot.

Note: If $curl\ \vec{A} = 0$ then vector field $\vec{A}$ is called Lamellar field.

Gauss' Divergence Theorem:

According to this theorem the volume integral of divergence of a vector field $\vec{A}$ over a volume V is equal to the surface integral of that vector field $\vec{A}$ taken over the surface S which enclosed that volume V. i.e.

$$\iiint_V (div\,\vec{A})dv = \iint_S \vec{A}.\overrightarrow{da}$$



Figure(12): Gauss' Divergence thorem

Consider a volume V enclosed by a surface S this volume can be divided into small elements of volumes $V_1, V_2 \ldots \ldots V_n$ which are enclosed by the elementary surface $S_1, S_2 \ldots \ldots \ldots S_n$ respectively. By definition the flux of a vector field $\vec{A}$ diverging out of the i^{th} element is

$$(div\,\vec{A})_i = \frac{\iint_{S_i} \vec{A}.\overrightarrow{da_i}}{V_i}$$

$$(div\,\vec{A})_i . V_i = \iint_{S_i} \vec{A}.\overrightarrow{da_i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

On LHS of equation we add the quantity $(div\,\vec{A})_i . V_i$ for each element $V_1, V_2 \ldots \ldots V_n$

$$\sum_{i=1}^{N} (div\,\vec{A})_i . V_i = \iiint_V (div\,\vec{A})\,dV$$

On RHS of equation (1) if we add the quantity $\iint_{S_i} \vec{A}.\overrightarrow{da_i}$ for each $S_1, S_2 \ldots \ldots \ldots S_n$ we get the terms only on the outer surface S

Sum comes out to be

$$\sum_{i=1}^{N} \iint_{Si} \vec{A}.\overrightarrow{da_i} = \iint_S \vec{A}.\overrightarrow{da}$$

So putting these values in equation (1) we get

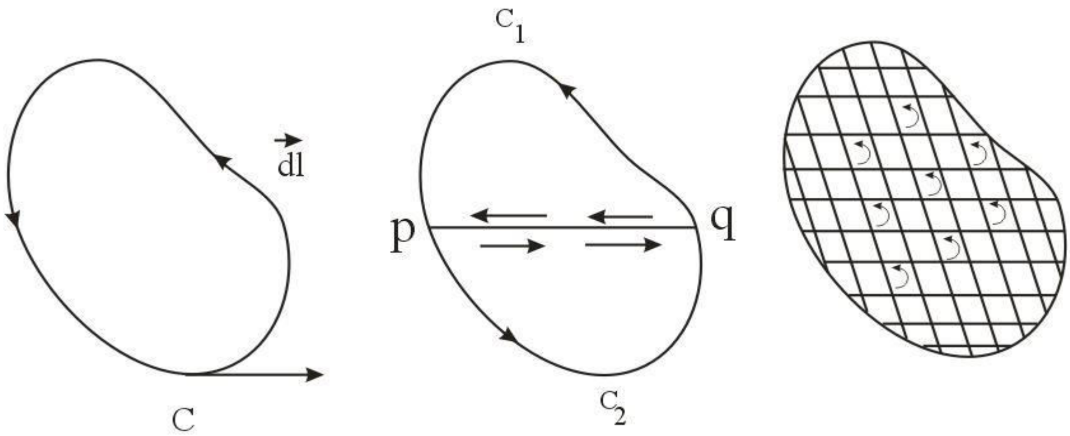
So $$\iiint_V (div\,\vec{A})\,dV = \iint_S \vec{A}.\overrightarrow{da}$$

This is the Gauss' divergence theorem.

Stokes theorem:

According to this theorem, the line integral of a vector field $\vec{A}$ along the boundary of a closed curve C is equal to the surface integral of curl of that vector field when the surface integration is done over a surface S enclosed by the boundary C i.e.

$$\oint_c \vec{A} \cdot \vec{dl} = \iint_S curl \vec{A} \cdot \vec{da}$$



Figure(13): Stokes theorem

Consider a vector $\vec{A}$ which is a function of position. We are to find the line integral $\oint_c \vec{A} \cdot \vec{dl}$ along the boundary of a closed curve C. If we divide the area enclosed by the curve C in two parts by a line pq, we get two closed curve C_1 and C_2. The line integral of vector $\vec{A}$ along the boundary of C will be equal to the sum of integral of $\vec{A}$ along $C_1 pq C_1$ and $C_2 qp C_2$

$$\oint_c \vec{A} \cdot \vec{dl} = \oint_{c_1} \vec{A} \cdot \vec{dl} + \oint_{c_2} \vec{A} \cdot \vec{dl}$$

Similarly if we divide the area enclosed by the curve C in small element of area $da_1\ da_2 \dots \dots \dots$ by the curve $C_1, C_2 \dots \dots \dots$ As shown in the figure. Then the sum of line integrals along the boundary of these curves $C_1, C_2 \dots \dots$ (taken anticlockwise) will be

$$\oint_c \vec{A} \cdot \vec{dl} = \sum \oint_{c_n} \vec{A} \cdot \vec{dl}$$

By the definition of curl, we have

$$curl\,\vec{A} = \frac{\oint_{c_n} \vec{A}.\overrightarrow{dl}}{\vec{d}_{a_n}}$$

$$curl\,\vec{A}.\,\vec{d}_{a_n} = \oint_{c_n} \vec{A}.\overrightarrow{dl}$$

$$\oint_{c} \vec{A}.\overrightarrow{dl} = \sum curl\,\vec{A}.\overrightarrow{da_n} = \iint_S curl\,\vec{A}.\overrightarrow{da}$$

$$\oint_{c} \vec{A}.\overrightarrow{dl} = \iint_S curl\,\vec{A}.\overrightarrow{da}$$

Gauss Law

According to this law, the net electric flux through any closed surface is $\frac{1}{\varepsilon_0}$ times of the total charge present inside it.

$$\phi = \frac{1}{\varepsilon_0} Q \qquad \ldots\ldots\ldots\ldots\ldots \qquad (1)$$

But by the definition of electric flux

$$\Rightarrow \qquad \phi = \iint_S \vec{E}.\overrightarrow{ds} \qquad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \qquad (2)$$

So by equation (1) and (2)

so
$$\iint_S \vec{E}.\overrightarrow{ds} = \frac{Q}{\varepsilon_0}$$

This is the integral form of Gauss' law.

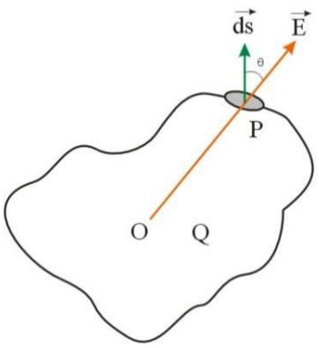
Proof:

Case1:

When the charge lies inside the arbitrary closed surface.

Let charge Q lies inside the arbitrary surface at point O

Now let us consider an infinitesimal area $\overrightarrow{ds}$ on this surface which contain the point P, the direction of the area vector $\overrightarrow{ds}$ is perpendicular to the surface and electric field



Figure(14): Gauss Law

$\vec{E}$ makes an angle θ with $\overrightarrow{ds}$ then electric field will be given as-

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3)$$

Now the flux emerging out of the surface area $\overrightarrow{ds}$ will be

$$d\phi = \vec{E}.\overrightarrow{ds}$$

$$\Rightarrow \quad d\phi = E\, ds \cos\theta$$

Where θ is the angle between $\vec{E}$ and $\overrightarrow{ds}$

So putting the value of $\vec{E}$ we get

$$d\phi = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} ds \cos\theta$$

$$\Rightarrow \quad d\phi = \frac{Q}{4\pi\varepsilon_0}\frac{ds \cos}{r^2}\theta$$

But $\frac{ds \cos}{r^2} \overset{\theta}{=} d\omega$ i.e. solid angle

$$d\phi = \frac{Q}{4\pi\varepsilon_0} d\omega$$

Now total flux

$$\phi = \int \frac{Q}{4\pi\varepsilon_0} d\omega$$

$$\Rightarrow \quad \phi = \frac{Q}{4\pi\varepsilon_0}\int d\omega$$

But $\int d\omega = 4\pi$

$$\phi = \frac{Q}{4\pi\varepsilon_0} 4\pi$$

$$\Rightarrow \quad \phi = \frac{Q}{\varepsilon_0}$$

Case 2:

When the charge lies outside the closed surface then the flux entering and leaving the surface area will be equal and opposite then $\phi = 0$

Gauss law in the differential form (Poisson's equation and Laplace's equation)

If the charge is continuous distributed over the volume and charge density is ρ

then $$Q = \iiint_V \rho \, dV$$

Now by Gauss theorem the flux emerging out of this surface which enclosed volume V

$$\iint_S \vec{E} \cdot \overrightarrow{ds} = \frac{1}{\varepsilon_0} \iiint_V \rho \, dV \quad \dots \dots \dots \dots \dots \dots \quad (1)$$

By Gauss divergence theorem

$$\iint_S \vec{E} \cdot \overrightarrow{ds} = \iiint_V div\,\vec{E}\, dV \quad \dots \dots \dots \dots \dots \dots \quad (2)$$

By equation (1) and (2)

$$\Rightarrow \quad \iiint_V div\,\vec{E}\, dV = \frac{1}{\varepsilon_0} \iiint_V \rho \, dV$$

$$\Rightarrow \quad \iiint_V \left(div\,\vec{E} - \frac{\rho}{\varepsilon_0} \right) dV = 0$$

But as we know that $dV \neq 0$

So $$div\,\vec{E} - \frac{\rho}{\varepsilon_0} = 0$$

$$\Rightarrow \quad div\,\vec{E} = \frac{\rho}{\varepsilon_0} \quad \dots \dots \dots \dots \dots \dots \quad (3)$$

This is the differential form of Gauss' law and also known as Poisson's equation

Now if we consider the charge less volume then $\rho = 0$

So $$div\,\vec{E} = 0 \quad \dots \dots \dots \dots \dots \quad (4)$$

This equation is Laplace equation.

Again by equation (3)

$$div\,\vec{E} = \frac{\rho}{\varepsilon_0}$$

We know that $\vec{E} = \; -gard\,V$

So

$$div\,(-grad\,V) = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \quad -\vec{\nabla}.\,(\vec{\nabla}V) = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \quad \nabla^2\,V = -\frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon_0}$$

Gauss law (in Presence of dielectrics):

The Gauss' law relates the electric flux and charge. The theorem states that the net electric flux across an arbitrary closed surface drown in an electric field is equal to $\frac{1}{\varepsilon_o}$ times the total charge enclosed by the surface. Now we want to extend this theorem for a region containing free charge embedded in dielectric.

In figure the dotted surface S in an imaginary closed surface drown in a dielectric medium. There is certain amount of free charge Q in the volume bounded by surface. Let us assume that free charge exists on the surface of three conductors in amount $q_1,\ q_2\ q_3 \ldots$ In a dielectric there also exits certain amount of polarisation (bound) charge Q_p.

Hence by Gauss' theorem

$$\iint_S \vec{E}\,.\,\vec{da} = \frac{1}{\varepsilon_0}(Q' + Q_p) \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Where $Q = q_1 + q_2 + q_3$ is the total free charge and Q_p is the polarisation (bound) charge by

$$Q_p = \iint_{S_1+S_2+S_3} \vec{P}.\,\vec{da} + \iiint_V (-divP)dV \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Here V is the volume of the dielectric enclosed by S. As there is no boundary of the dielectric at S, therefore the surface integral in equation (2) does not contain a contribution from S. If we transform volume integral in (2) into surface integral by means of Gauss divergence theorem, we must include contribution from all surface bounding V, namely $S,\ S_1,\ S_2$ and S_3 i..e.

$$\int_V (-divP)dV = \left[\iint_{S_1+S_2+S_3} \vec{P}.\,\vec{da} + \iiint_V (-divP)dV\right]$$

Using above equation, we note that

$$Q_p = \iint_{S_1+S_2+S_3} \vec{P}.\overrightarrow{da} \qquad \text{..................(3)}$$

Substituting this value in (1)

We get

$$\iint_S \vec{E}.\overrightarrow{da} = \frac{Q}{\varepsilon_0} - \frac{1}{\varepsilon_0}\iint_S \vec{P}.\overrightarrow{da}$$

$$\iint_S \left(\vec{E} + \frac{\vec{P}}{\varepsilon_0}\right).\overrightarrow{da} = \frac{Q}{\varepsilon_0}$$

Multiplying through by ε_0

$$\iint_S (\varepsilon_0\vec{E} + \vec{P}).\overrightarrow{da} = Q \qquad \text{..................(4)}$$

This equation states that the flux of the vector $(\varepsilon_0\vec{E} + \vec{P})$ through a closed surface is equal to the total free charge enclosed by the surface. This vector quantity is named as electric displacement $\vec{D}$ i.e.

$$\vec{D} = \varepsilon_0\vec{E} + \vec{P} \qquad \text{.........................(5)}$$

Evidently electric displacement $\vec{D}$ has the same unit as $\vec{P}$. i.e. charge per unit area.

In terms of electric displacement vector $\vec{D}$, equation (4) becomes

$$\iint_S \vec{D}.\overrightarrow{da} = Q \qquad \text{.........................(6)}$$

i.e. the flux of electric displacement vector across an arbitrary closed surface is equal to the total free charge enclosed by the surface.

This result is usually referred to as Gauss' theorem in dielectric.

If we consider into a large number of infinitesimal volume elements, then Gauss' t expressed as

$$\iint_S \vec{D}.\overrightarrow{da} = \iiint_V \rho\, dV \qquad \text{.........................(7)}$$

Where ρ is the charge density at a point within volume element dV such that $dV \rightarrow 0$.

$$\iiint_V div\vec{D}\, dV = \iiint_V \rho\, dV$$

$$\iiint_V (div\,\vec{D} - \rho).\,dV = 0$$

Volume is arbitrary, therefor we get

$$div\,\vec{D} - \rho = 0$$

$$div\,\vec{D} \;=\; \rho$$

This result is called differential form of Gauss' theorem in a dielectric.

The main advantage of this method is that the total electrostatic field at each point in the dielectric medium may be expressed as the sum of parts

$$E(x,\; y, \;=\; \frac{1}{\varepsilon_0}\vec{D}(x,\; y,) - z\frac{1}{\varepsilon_0}P(x,\; y,\; z) \qquad \text{.........................(8)}$$

Where the first term $\frac{1}{\varepsilon_0}\vec{D}$ is related to free charge density through the divergence and the second theorem $\frac{1}{\varepsilon_0}P$ is proportional to the polarisation of the medium. In vacuum $\rho \;= s0 \vec{E} = \frac{\vec{D}}{\varepsilon_0}$

Electric Polarization($\boldsymbol{P}$)

When a dielectric is placed in any external electric field then the dielectric gets polarized and induced electric dipole moment is produced which is proportional to the external applied electric field. Now if there are n number of dipoles induced in per unit volume of dielectric then total polarization will be-

$$\vec{P} \;=\; n\,\overrightarrow{P_{in}} \qquad \ldots \;\ldots \;\ldots \;\ldots \;\ldots \;\ldots \quad (1)$$

But

$$\overrightarrow{P_{in}} \;\propto\; \overrightarrow{E_0}$$

So

$$\overrightarrow{P_{in}} \;=\; \varepsilon_0\,\alpha_{in}\overrightarrow{E_0}$$

Putting this value in equation (1) we get

$$\Rightarrow \qquad \vec{P} \;=\; n\,\varepsilon_0\,\alpha_{in}\overrightarrow{E_0}$$

It is clear from the above equation that the direction of polarization is in the direction of the applied external electric field. And the unit is $coulomb/meter^2$

Electric displacement(D)

We know that the value of electric field depends on the nature of the material, so to study the dielectric we need such a quantity which does not depends on the nature of the material and this quantity is known as electric displacement vector $\vec{D}$.

Both $\vec{E}$ and $\vec{D}$ are same except that we define $\vec{E}$ by a force in a charge placed at a point while the displacement vector is measure by the displacement flux per unit area at that point.

$$\iiint_S \vec{D}\cdot\vec{ds} = q$$

Or

$$D = \frac{q}{A}$$

$$\Rightarrow \qquad D = \sigma$$

Where σ is the surface charge density.

Relation between $\vec{E}$ and $\vec{D}$

We know that the Gauss law is given as-

$$\iint_s \vec{E}\cdot\vec{ds} = \frac{q}{\varepsilon}$$

Where ε is the permittivity of the dielectric medium

$$\Rightarrow \qquad \vec{E} = \frac{1}{\varepsilon}\cdot\frac{q}{A}$$

But $\frac{q}{A} = D$ So we have $\vec{E} = \frac{1}{\varepsilon}\vec{D} \Rightarrow \vec{D} = \varepsilon\vec{E}$

$$\Rightarrow \vec{D} = \varepsilon_0 K\vec{E}$$

$$\therefore \varepsilon = \varepsilon_0 K$$

Where ε_0 is the permittivity of the free space

Current:

Current for study current

$$I = \frac{q}{t}$$

If the charge passing per unit time is not constant, then the current at any instant will be given as

$$I = \frac{dq}{dt}$$

Current density:

$$\vec{J} = \frac{dI}{\overrightarrow{da}}$$

$$dI = \vec{J}.\overrightarrow{da}$$

$$I = \int \vec{J}.\overrightarrow{da} = \frac{dq}{dt}$$

From the above equation we can see that the current is the flux of current density as

$$\phi = \int \vec{E}\,\overrightarrow{da}$$

Its SI unit is

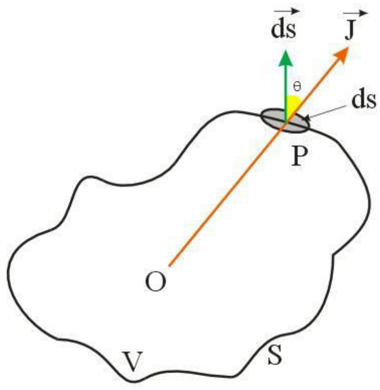
$$\frac{ampear}{meter^2}$$

Equation of continuity:

The law of conservation of charge is called the equation of the

$$I = \iint_s \vec{J} \cdot \vec{ds}$$

For steady current charge does not stay at any place, so the current will be constant for all the places.



Figure(17): Flux of current

$$\Rightarrow \qquad I = \iint_s \vec{J} \cdot \vec{ds} = k$$

By divergence theorem

$$\Rightarrow \qquad \iint_s \vec{J} \cdot \vec{ds} = \iiint_V div\vec{v}.J \; dV$$

So

$$\iiint_V div\vec{v}.J \; dV = k$$

On differentiating we get

$$div\,\vec{J} = 0$$

This is the equation of continuity for study current.

Now if current is not stationary i.e. if the current is the function of the time and position

then

$$I = \iint_s \vec{J} \cdot \vec{ds} = \frac{dq}{dt}$$

Here negative sign shows that the charge is reduced with respect to time.

But if ρ is the charge per unit volume then-

$$q = \iiint_V \rho. \; dV$$

So

$$\iint_s \vec{J} \cdot \vec{ds} = -\frac{d}{dt} \iiint_V \rho. \; dV$$

$$\Rightarrow \qquad \iiint_V div\vec{v}.J \; dV = -\frac{d}{dt} \iiint_V \rho. \; dV$$

$$\Rightarrow \qquad \iiint_V \left(div\vec{J} + \frac{d\rho}{dt} \right) dV = 0$$

$$\Rightarrow \qquad div\,\vec{J} = -\frac{d\rho}{dt}$$

This is the equation of continuity for time varying current.

Maxwell's equations

James Clerk Maxwell took a set of known experimental laws (Faraday's Law, Ampere's Law) and unified them into a symmetric coherent set of Equations known as Maxwell's Equations. These equations are nothing but the relation between electric field and magnetic field in terms of divergence and curl.

S.N.	Name	Integral form	Differential form
1	Gauss' Law for electricity	$\iint_s \vec{E}\,\vec{ds} = \frac{1}{\varepsilon_0}\iiint_V \rho\, dV$	$div\,\vec{E} = \frac{\rho}{\varepsilon_0}$
2	Gauss' law for magnetism	$\iint_s \vec{B}\,\vec{ds} = 0$	$div\,\vec{B} = 0$
3	Faraday's Law of induction	$\oint_c \vec{E}\,\vec{dl} = \frac{\partial}{\partial t}\iint_s \vec{B}\,\vec{ds}$	$curl\,\vec{E} = \frac{\partial\vec{B}}{\partial t}$
4	Ampere's law	$\oint_c \vec{E}\,\vec{dl} = \iint_s \vec{J}\,\vec{ds} + \frac{\partial}{\partial t}\iint_s \vec{D}\,\vec{ds}$	$curl\,\vec{B} = \mu_0\left(\vec{J} + \varepsilon_0\frac{\partial\vec{E}}{\partial t}\right)$

Maxwell's first equation (Gauss' law in electric):

Let us consider a volume V which is enclosed in a surface S, then by Gauss' law the electric flux is given as

$$\iint_s \vec{E}\,\vec{ds} = \frac{1}{\varepsilon_0} q \qquad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ . \quad (1)$$

Where q is the totat charge enclosed in the volume V

Now if ρ is the volume charge density then

$$q = \iiint_v \rho\, dV \qquad \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ .. \quad (2)$$

By equation (1) and (2)

$$\Rightarrow \quad \iint_s \vec{E}\,\vec{ds} = \frac{1}{\varepsilon_0}\iiint_V \rho\, dV$$

This is the integral form of Maxwell's equation.

By Gauss' divergence theorem

$$\Rightarrow \quad \iint_s \vec{E}\,\vec{ds} = \iiint_V div\,\vec{E}\,dV$$

So by applying this on above equation we get

$$\Rightarrow \quad \iiint_V div\,\vec{E}\,dV = \frac{1}{\varepsilon_0}\iiint_V \rho\, dV$$

$$\Rightarrow \quad \iiint_V \left(div\,\vec{E} - \frac{\rho}{\varepsilon_0}\right) dV = 0$$

But $dV \neq 0$ so

$$\Rightarrow \quad div\,\vec{E} - \frac{\rho}{\varepsilon_0} = 0$$

$$\Rightarrow \quad div\,\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \quad div\,\varepsilon_0\vec{E} = \rho$$

$$\Rightarrow \quad div\,\vec{D} = \rho \qquad [\because \vec{D} = \varepsilon_0\vec{E}]$$

Maxwell's second equation (Gauss' law in magnetism):

Since the magnetic lines of forces are closed curves so the magnetic flux entering any orbitri surface should be equal to leaving it

mathematically

$$\Rightarrow \quad \iint_s \vec{B}\,\vec{ds} = 0 \qquad \dots \dots \dots \dots \quad (1)$$

This is integral form of Maxwell's second equation.

Now by Gauss' divergence theorem

$$\Rightarrow \quad \iint_s \vec{B}\,\vec{ds} = \iiint_V div\,\vec{B}\,dV$$

So equation (1) can be written as-

$$\Rightarrow \quad \iiint_V div\,\vec{B}\,dV = 0$$

As $dV \neq s\!o0$

$$\Rightarrow \qquad div\,\vec{B} = 0$$

Maxwell's third equation (Faraday's law):

According to Faraday's law of electromagnetic induction if the magnetic flux linked with a closed circuit changes with time then a emf is induced in the close circuit which is known as induced emf the direction of the induced emf will be such as it oppose the change in the magnetic flux. It is given as

$$\Rightarrow \qquad e = -\frac{d\phi}{dt} \qquad \dots \dots \dots \dots \dots \dots \quad (1)$$

But by Gauss' theorem we know that

$$\Rightarrow \qquad \phi = \iint_s \vec{B}\,\vec{dl}$$

$$\text{So} \qquad e = -\frac{\partial}{\partial t}\iint_s \vec{B}\,\vec{ds}$$

Now if $\vec{E}$ is the electric field produced due to the change in the magnetic flux then the induced emf will be equal to the line integral of $\vec{E}$ along the circuit. i.e.

$$\Rightarrow \qquad e = \oint_c \vec{E}.\,\vec{dl} \qquad \dots \dots \dots \dots \dots \quad (2)$$

By equation (1) and (2)

$$\Rightarrow \qquad \oint_c \vec{E}\,\vec{d} = -\frac{\partial}{\partial t}\iint_s \vec{B}\,\vec{ds}$$

$$\Rightarrow \qquad \oint_c \vec{E}\,\vec{d} = -\iint_s \frac{\partial \vec{B}}{\partial t}\,\vec{ds} \qquad \dots \dots \dots \dots \dots \quad (3)$$

Now Stokes' theorem

$$\Rightarrow \qquad \oint_c \vec{E}\,\vec{d} = \iint_s curl\,\vec{E}\,\vec{ds}$$

Applying this to the above equation, we get

$$\iint_s curl\,\vec{E}\,\overrightarrow{ds} = -\iint_s \frac{\partial \vec{B}}{\partial t}\,\overrightarrow{ds}$$

$$\Rightarrow \qquad \iint_s \left(curl\,\vec{E} + \frac{\partial \vec{B}}{\partial t}\right)\overrightarrow{ds} = 0$$

As $\overrightarrow{ds} \neq 0$

So

$$curl\,\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

$$\Rightarrow \qquad curl\,\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Maxwell's fourth equation (Maxwell's correction in Ampere's law)

Ampere's Law is given as

$$\Rightarrow \qquad curl\,\vec{B} = \mu_0\,\vec{J}$$

This equation is true only for time independent electric field and to correct this equation for time varying field a term must be added

$$\Rightarrow \qquad curl\,\vec{B} = \mu_0(\vec{J} + \vec{J}_d) \quad \dots \dots \dots \dots \dots \dots \quad (1)$$

Taking $divergence$ of both side and for simplicity writing $\vec{J}_{free}$ as $\vec{J}$

$$\Rightarrow \qquad div\,(curl\,\vec{B}) = \mu_0\,div(\vec{J} + \vec{J}_d)$$

But divergence of curl of any quantity is always zero so $div\,(curl\,\vec{B}) = 0$

$$\text{Then} \qquad \mu_0\,div(\vec{J} + \vec{J}_d) = 0 \quad \dots\dots\dots\dots \quad (2)$$

$$\Rightarrow \qquad div\,\vec{J} = -div\,\vec{J}_d \quad \dots \dots \dots \dots \dots \dots \quad (3)$$

But by the equation of continuity

$$\Rightarrow \qquad div\,\vec{J} = -\frac{\partial \rho}{\partial t} \quad \dots \dots \dots \dots \dots \dots \quad (4)$$

And by Maxwell's first equation

$$div\,\vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\Rightarrow \quad \rho = \varepsilon_0\, div\,\vec{E} \quad \dots \dots \dots \dots \dots \dots \quad (5)$$

By (4) and (5)

$$\Rightarrow \quad div\,\vec{J} = -\frac{\partial}{\partial t}\left(\varepsilon_0\, div\,\vec{E}\right)$$

$$\Rightarrow \quad div\,\vec{J} = -\frac{\partial}{\partial t}\left(div\,\vec{D}\right) \quad \dots \dots \dots \dots \dots . \quad (6)$$

Again by (3) and (6)

$$\Rightarrow \quad -div\,\vec{J}_d = -\frac{\partial}{\partial t}\left(div\,\vec{D}\right)$$

$$\Rightarrow \quad div\,\vec{J}_d = \frac{\partial}{\partial t}\left(div\,\vec{D}\right)$$

$$\Rightarrow \quad div\,\vec{J}_d = div\left(\frac{\partial}{\partial t}\vec{D}\right)$$

$$\Rightarrow \quad \vec{J}_d = \frac{\partial \vec{D}}{\partial t}$$

Putting this value in Ampere’s law we get

$$curl\,\vec{B} = \mu_0\left(\vec{J} + \frac{\partial \vec{D}}{\partial t}\right)$$

This is Maxwell’s fourth equation.

For vacuum $\vec{B} = \mu_0 H$ and $D = \varepsilon_0 E$

So

$$\mu_0\, curl\,\vec{H} = \mu_0\left(\vec{J} + \varepsilon_0\frac{\partial \vec{E}}{\partial t}\right)$$

$$\Rightarrow \quad curl\,\vec{H} = \left(\vec{J} + \varepsilon_0\frac{\partial \vec{E}}{\partial t}\right)$$