

UNIT-I

Virtual work and Energy Principles: Principles of Virtual work applied to deformable bodies, strain energy and complementary energy, Energy theorems, Maxwell's Reciprocal theorem, Analysis of Pin-Jointed frames for static loads.

Energy Methods in Structural Analysis**Virtual Work****Introduction**

. From Castigliano's theorem it follows that for the statically determinate structure; the partial derivative of strain energy with respect to external force is equal to the displacement in the direction of that load. In this lesson, the principle of virtual work is discussed. As compared to other methods, virtual work methods are the most direct methods for calculating deflections in statically determinate and indeterminate structures. This principle can be applied to both linear and nonlinear structures. The principle of virtual work as applied to deformable structure is an extension of the virtual work for rigid bodies. This may be stated as: if a rigid body is in equilibrium under the action of a system of forces and if it continues to remain in equilibrium if the body is given a small (virtual) displacement, then the virtual work done by the F–F–system of forces as 'it rides' along these virtual displacements is zero.

Complementary Strain Energy:

Consider the stress strain diagram as shown Fig 1.1. The area enclosed by the inclined line and the vertical axis is called the complementary strain energy. For a linearly elastic material the complementary strain energy and elastic strain energy are the same.

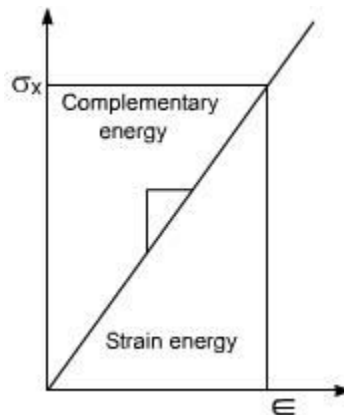


Figure 1.1: Stress strain diagram.

Let us consider elastic nonlinear prismatic bar subjected to an axial load. The resulting stress strain plot is as shown.

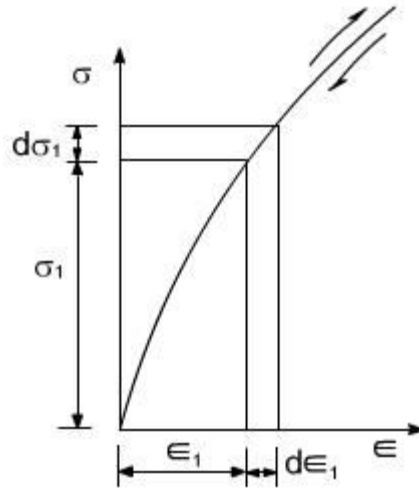


Figure 1.2: The resulting Stress strain diagram.

The new term complementary work is defined as follows

$$W^* = \int_0^P \delta_1 dP_1$$

we also know

$$W^* + W = P\delta$$

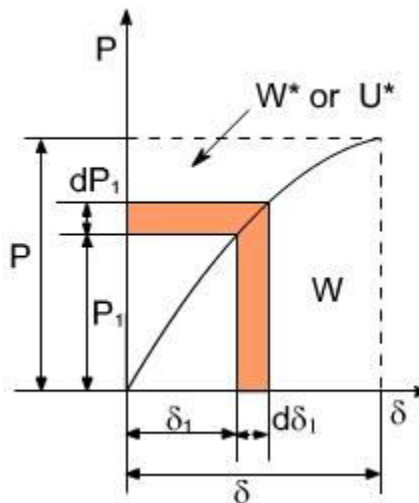


Figure 1.3: The resulting Stress strain diagram.

So in geometric sense the work W^* is the complement of the work ' W ' because it completes rectangle as shown in the above figure

Complementary Energy

$$U^* = W^* = \int_0^P \delta_1 dP$$

Likewise the complementary energy density u^* is obtained by considering a volume element subjected to the stress s_1 and $\hat{\epsilon}_1$, in a manner analogous to that used in defining the strain energy density. Thus

$$U^* = \int_0^{\sigma} \epsilon_1 d\sigma_1$$

The complementary energy density is equal to the area between the stress strain curve and the stress axis. The total complementary energy of the bar may be obtained from u^* by integration

$$U^* = \int dV$$

Sometimes the complementary energy is also called the stress energy. Complementary Energy is expressed in terms of the load and that the strain energy is expressed in terms of the displacement.

Castigliano's Theorem: Strain energy techniques are frequently used to analyze the deflection of beam and structures. Castigliano's theorem were developed by the Italian engineer Alberto Castigliano in the year 1873, these theorems are applicable to any structure for which the force deformation relations are linear

Castigliano's Theorem:

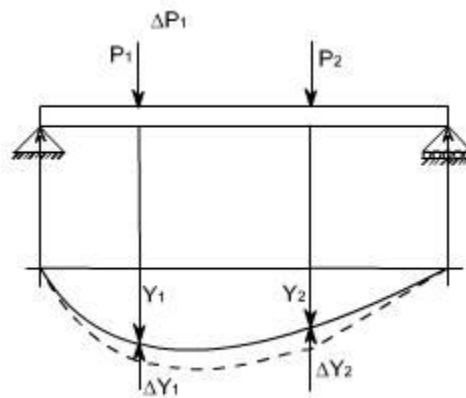


Figure 1.4: The loaded beam.

Consider a loaded beam as shown in figure 1.4

Let the two Loads P_1 and P_2 produce deflections Y_1 and Y_2 respectively strain energy in the beam is equal to the work done by the forces.

$$U = \frac{1}{2} P_1 Y_1 + \frac{1}{2} P_2 Y_2 \quad \dots(1)$$

Let the Load P_1 be increased by an amount DP_1 .

Let DP_1 and DP_2 be the corresponding changes in deflection due to change in load to DP_1 .

Now the increase in strain energy $\Delta U = \frac{1}{2} \Delta P_1 \Delta Y_1 + P_1 \Delta Y_1 + P_2 \Delta Y_2 \quad \dots(2)$

Suppose the increment in load is applied first followed by P_1 and P_2 then the resulting strain energy is

$$U + \Delta U = \frac{1}{2} \Delta P_1 \Delta Y_1 + \Delta P_1 Y_1 + P_2 \Delta Y_2 + \frac{1}{2} P_1 Y_1 + \frac{1}{2} P_2 Y_2 \quad \dots(3)$$

Since the resultant strain energy is independent of order loading,

Combining equation 1, 2 and 3. One can obtain

$$\Delta P_1 Y_1 = P_1 \Delta Y_1 + P_2 \Delta Y_2 \quad \dots(4)$$

equations (2) and (4) can be combined to obtain

$$\frac{\Delta U}{\Delta P_1} = Y_1 + \frac{1}{2} \Delta Y_1 \quad \dots(5)$$

or upon taking the limit as ΔP_1 approaches zero [Partial derivative are used because the strain energy is a function of both P_1 and P_2]

$$\frac{\partial U}{\partial P} = Y_1 \quad \dots(6)$$

For a general case there may be number of loads, therefore, the equation (6) can be written as

$$\frac{\partial U}{\partial P_i} = Y_i \quad \dots(7)$$

The above equation is castigation's theorem:

The statement of this theorem can be put forth as follows; if the strain energy of a linearly elastic structure is expressed in terms of the system of external loads. The partial derivative of strain energy with respect to a concentrated external load is the deflection of the structure at the point of application and in the direction of that load.

In a similar fashion, Castigliano's theorem can also be valid for applied moments and resulting rotations of the structure

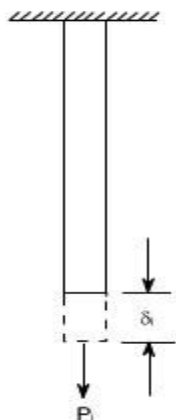
$$\frac{\partial U}{\partial M_i} = \theta_i \quad \dots(8)$$

Where

M_i = applied moment

θ_i = resulting rotation

Castigliano's First Theorem:



In similar fashion as discussed in previous section suppose the displacement of the structure are changed by a small amount dd_i . While all other displacements are held constant the increase in strain energy can be expressed as

$$dU = \frac{\partial U}{\partial \delta_i} d\delta_i \quad \dots (9)$$

Where

$\partial U / \partial \delta_i$ is the rate of change of the strain energy w.r.t δ_i .

It may be seen that, when the displacement δ_i is increased by the small amount dd ; work done by the corresponding force only since other displacements are not changed.

The work which is equal to $P_i dd_i$ is equal to increase in strain energy stored in the structure

$$dU = P_i d\delta_i$$

By rearranging the above expression, the Castigliano's first theorem becomes

$$P_i = \frac{dU}{d\delta_i}$$

The above relation states that the partial derivative of strain energy w.r.t. any displacement δ_i is equal to the corresponding force P_i provided that the strain is expressed as a function of the displacements.

Maxwell-Betti Law of Reciprocal Deflections

Maxwell-Betti Law of real work is a basic theorem in the structural analysis. Using this theorem, it will be established that the flexibility coefficients in compatibility equations, formulated to solve indeterminate structures by the flexibility method, form a symmetric matrix and this will reduce the number of deflection computations. The Maxwell-Betti law also has applications in the construction of influence lines diagrams for statically indeterminate structures. The Maxwell-Betti law, which applies to any stable elastic structure (a beam, truss, or frame, for example) on unyielding supports and at constant temperature, states:

The deflection of point A in direction 1 due to unit load at point B in direction 2 is equal in the magnitude to the deflection of point B in direction 2 produced by a unit load applied at A in direction 1.

The Figure 4.31 explains the Maxwell-Betti Law of reciprocal displacements in which, the displacement Δ_{AB} is equal to the displacement.

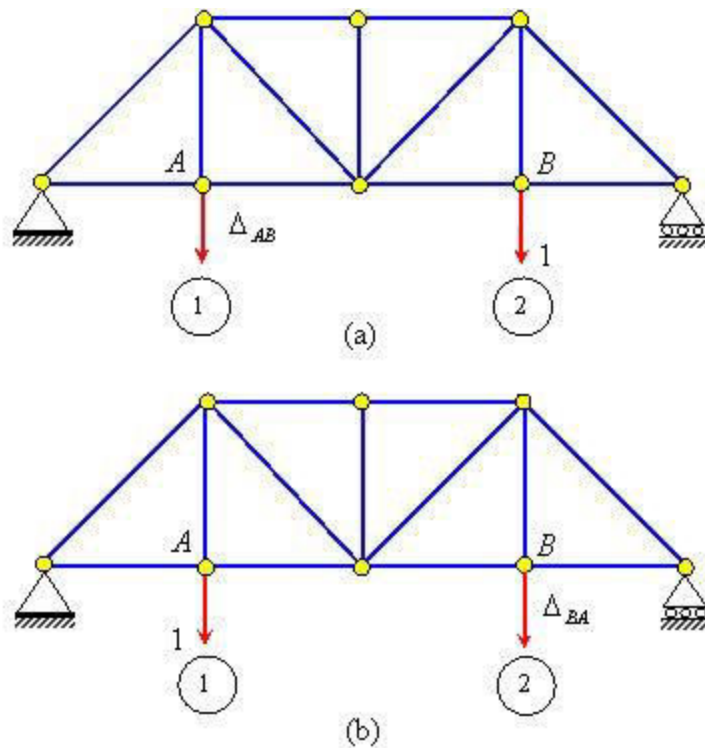


Figure 4.31 Illustration of Maxwell-Betti Law (directions 1 and 2 are shown by circle)

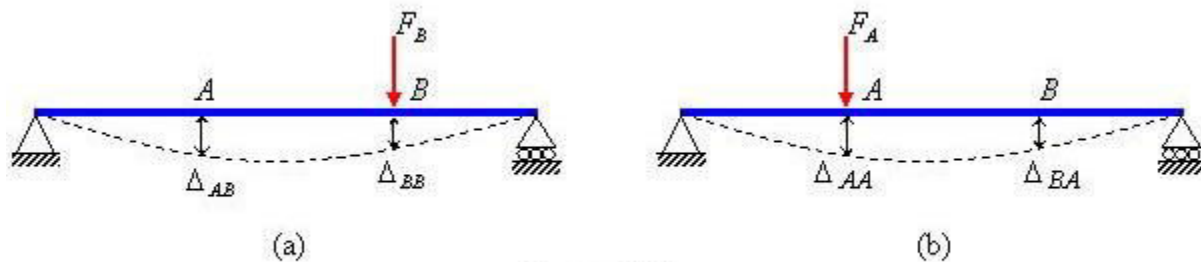


Figure 4.32

In order to prove the reciprocal theorem, consider the simple beams shown in Figure 4.32. Let a vertical force F_B at point B produces a vertical deflection Δ_{AB} at point A and Δ_{BB} at point B as shown in Figure 4.32(a). Similarly, in Figure 4.32(b) the application of a vertical force F_A at point A produces a vertical deflections Δ_{AA} and Δ_{BA} at points A and B, respectively. Let us evaluate the total work done by the two forces F_A and F_B when they are applied in different order to the zero to their final value.

Case 1: F_B applied and followed by F_A

(a) Work done when F_B is gradually applied

$$W_B = \frac{1}{2} F_B \Delta_{BB}$$

(b) Work done when F_A is gradually applied with F_B in place

$$W_A = \frac{1}{2} F_A \Delta_{AA} + F_B \Delta_{BA}$$

Total work done by the two forces for case 1 is

$$\begin{aligned} W_1 &= W_B + W_A \\ &= \frac{1}{2} F_B \Delta_{BB} + \frac{1}{2} F_A \Delta_{AA} + F_B \Delta_{BA} \end{aligned}$$

Case2: F_A applied and followed by F_B

(c) Work done when F_A is gradually applied

$$W_A = \frac{1}{2} F_A \Delta_{AA}$$

(d) Work done when F_B is gradually applied with F_A in place

$$W_B = \frac{1}{2} F_B \Delta_{BB} + F_A \Delta_{AB}$$

Total work done by the two forces for case 2 is

$$\begin{aligned} W_2 &= W_B + W_A \\ &= \frac{1}{2} F_A \Delta_{AA} + \frac{1}{2} F_B \Delta_{BB} + F_A \Delta_{AB} \end{aligned}$$

Since the final deflected position of the beam produced by the two cases of loads is the same regardless of the order in which the loads are applied. The total work done by the forces is also the same regardless of the order in which the loads are applied. Thus, equating the total work of Cases 1 and 2 give

$$W_1 = W_2$$

$$\frac{1}{2} F_B \Delta_{BB} + \frac{1}{2} F_A \Delta_{AA} + F_B \Delta_{BA} = \frac{1}{2} F_A \Delta_{AA} + \frac{1}{2} F_B \Delta_{BB} + F_A \Delta_{AB}$$

$$F_B \Delta_{BA} = F_A \Delta_{AB}$$

If $F_A = F_B = 1$, the equation (4.31) depicts the statement of the Maxwell-Betti law i.e.

$$\Delta_{BA} = \Delta_{AB}$$

The Maxwell-Betti theorem also holds for rotations as well as rotation and linear displacement in beams and frames.

Example 4.21 Verify Maxwell-Betti law of reciprocal displacement for the direction 1 and 2 of the pin-jointed structure shown in Figure (a).

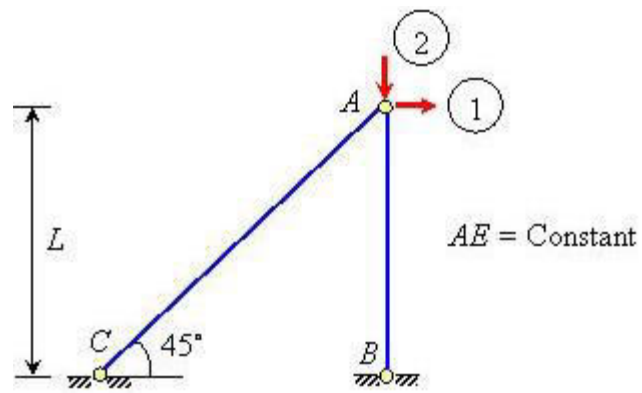


Figure 4.33(a)

Solution: Apply the forces P_1 and P_2 in the direction 1 and 2, respectively. The calculation of total strain energy in the system is given in Table 4.5.

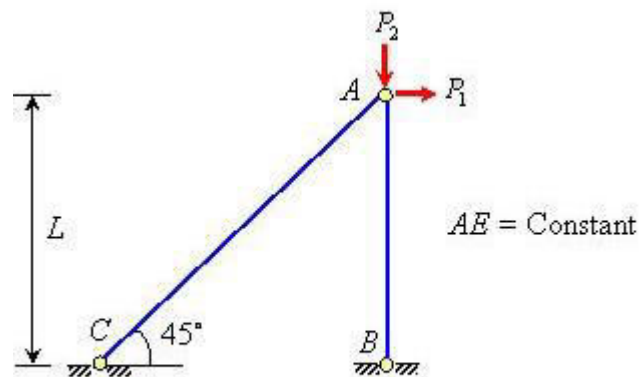


Figure 4.33(a)

Table 4.5

Membe	Length	Force P	$U = \frac{P^2 L}{2AE}$
AB	L	$P_1 + P_2$	$\frac{(P_1 + P_2)^2 L}{2AE}$
AC	$\sqrt{2} L$	P_1	$\frac{\sqrt{2} P_1^2 L}{AE}$

$$\sum ((P_1 + P_2)^2 + 2\sqrt{2}P_1^2)L / 2AE$$

$$\Delta_{21} = \left. \frac{\partial U}{\partial P_2} \right|_{P_1=1, P_2=0}$$

$$= (2(P_1 + P_2) + 0) \frac{L}{2AE} \Big|_{P_1=1, P_2=0}$$

$$= \frac{L}{AE}$$

$$\Delta_{12} = \left. \frac{\partial U}{\partial P_1} \right|_{P_1=0, P_2=1}$$

$$= (2(P_1 + P_2) + 4\sqrt{2}P_1) \frac{L}{2AE} \Big|_{P_1=0, P_2=1}$$

$$= \frac{L}{AE}$$

Since $\Delta_{12} = \Delta_{21}$, hence the Maxwell-Betti law of reciprocal displacement is proved.

Example: Verify Maxwell-Betti law of reciprocal displacement for the cantilever beam shown in Figure 4.34(a).

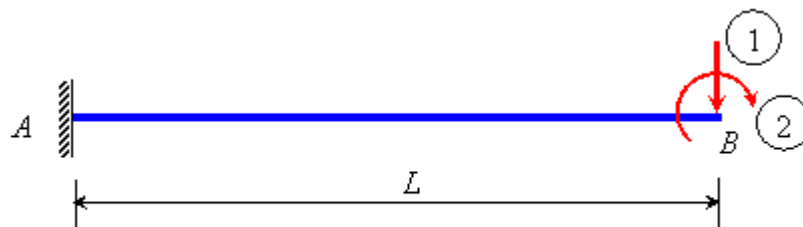


Figure 4.34(a)

Solution: Apply the forces P_1 and P_2 in the directions 1 and 2, respectively. The total strain energy stored is calculated below.

Consider any point X at a distance x from B .

$$M_x = -(P_1 x + P_2 L)$$

$$U = \int_0^L \frac{M_x^2 dx}{2EI}$$

$$= \frac{1}{2EI} \int_0^L (P_1 x + P_2 L)^2 dx$$

$$= \frac{1}{2EI} \left(\frac{P_1^2 L^3}{3} + P_1 P_2 L^2 + P_2^2 L \right)$$

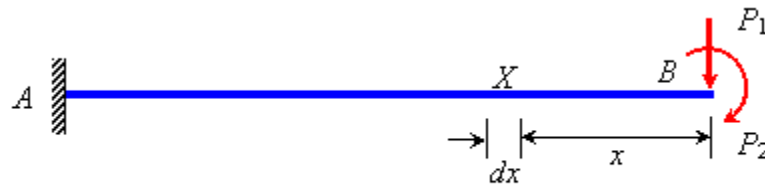


Figure 4.34(b)

$$\begin{aligned}\Delta_{12} &= \left. \frac{\partial U}{\partial P_1} \right|_{P_1=0, P_2=1} \\ &= \frac{1}{2EI} \left(\frac{2P_1 L^3}{3} + P_2 L^2 + 0 \right) \bigg|_{P_1=0, P_2=1} \\ &= \frac{L^2}{2EI}\end{aligned}$$

$$\begin{aligned}\Delta_{21} &= \left. \frac{\partial U}{\partial P_2} \right|_{P_1=1, P_2=0} \\ &= \frac{1}{2EI} (0 + P_1 L^2 + 2P_2 L) \bigg|_{P_1=1, P_2=0}\end{aligned}$$

$$= \frac{L^2}{2EI}$$

Since $\Delta_{12} = \Delta_{21}$, the Maxwell-Betti law of reciprocal displacement is proved.

Example: Verify Maxwell-Betti law of reciprocal displacement for the rigid-jointed plane frame with reference to marked direction as shown in Figure 4.35(a). EI is same for both members.

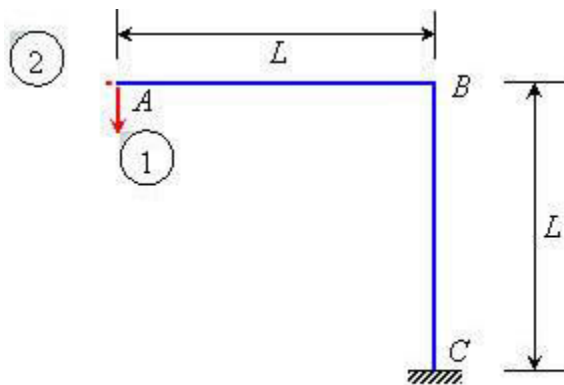


Figure 4.35(a)

Solution: Apply the forces P_1 and P_2 in the directions 1 and 2, respectively as shown in Figure (b).

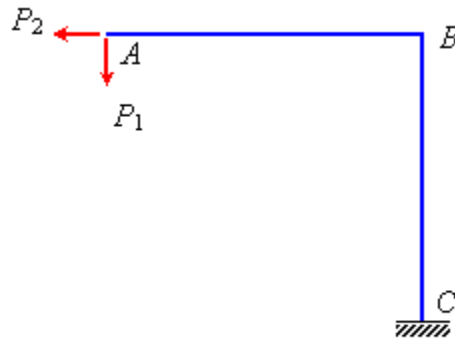


Figure 4.35(b)

Consider AB: (x measured from A)

$$M_x = -P_1 x$$

$$\begin{aligned} U_{AB} &= \int_0^L \frac{M_x^2 dx}{2EI} \\ &= \frac{1}{2EI} \int_0^L (-P_1 x)^2 dx \\ &= \frac{P_1^2 L^3}{6EI} \end{aligned}$$

Consider BC: (x measured from B)

$$\begin{aligned} M_x &= -P_1 L - P_2 x \\ U_{BC} &= \int_0^L \frac{M_x^2 dx}{2EI} \\ &= \frac{1}{2EI} \int_0^L (-P_1 L - P_2 x)^2 dx \\ &= \frac{P_1^2 L^3}{2EI} + \frac{P_1 P_2 L^3}{2EI} + \frac{P_2^2 L^3}{6EI} \end{aligned}$$

Thus

$$\begin{aligned} U &= U_{AB} + U_{BC} \\ &= \frac{P_1^2 L^3}{6EI} + \frac{P_1^2 L^3}{2EI} + \frac{P_1 P_2 L^3}{2EI} + \frac{P_2^2 L^3}{6EI} \\ &= \frac{L^3}{6EI} (4P_1^2 + 3P_1 P_2 + P_2^2) \end{aligned}$$

The displacement in the direction 1 due to unit load applied in 2 is

$$\begin{aligned}
 \Delta_{12} &= \frac{\partial U}{\partial P_1} \bigg|_{P_1=0, P_2=1} \\
 &= \frac{L^3}{6EI} (8P_1 + 3P_2 + 0) \bigg|_{P_1=0, P_2=1} \\
 &= \frac{L^3}{2EI}
 \end{aligned}$$

The displacement in the direction 2 due to unit load applied in 1 is

$$\begin{aligned}
 \Delta_{21} &= \frac{\partial U}{\partial P_2} \bigg|_{P_1=1, P_2=0} \\
 &= \frac{L^3}{6EI} (0 + 3P_1 + 2P_2) \bigg|_{P_1=1, P_2=0} \\
 &= \frac{L^3}{2EI}
 \end{aligned}$$

Since $\Delta_{12} = \Delta_{21}$, proves the Maxwell-Betti law of reciprocal displacements.

Unit II

Indeterminate Structures-I: Static and Kinematics indeterminacy, Analysis of Fixed and continuous beams by theorem of three moments, Effect of sinking and rotation of supports, Moment distribution method (without sway)

Three Moment Equation

The continuous beams are very common in the structural design and it is necessary to develop simplified force method known as *three moment equation* for their analysis. This equation is a relationship that exists between the moments at three points in continuous beam. The points are considered as three supports of the indeterminate beams. Consider three points on the beam marked as 1, 2 and 3 as shown in Figure (a). Let the bending moment at these points is M_1 , M_2 and M_3 , and the corresponding vertical displacement of these points are Δ_1 , Δ_2 and Δ_3 , respectively. Let L_1 and L_2 be the distance between points 1 – 2 and 2 – 3, respectively.

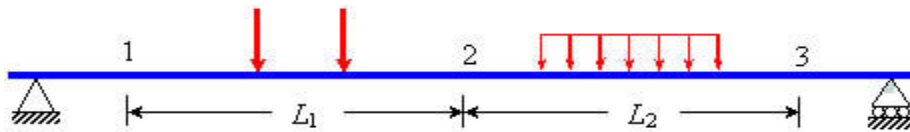


Figure 5.25(a)

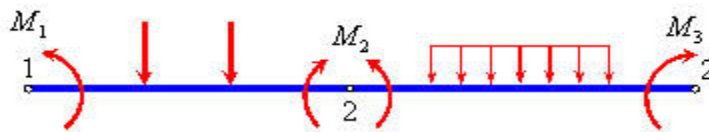


Figure 5.25(b)

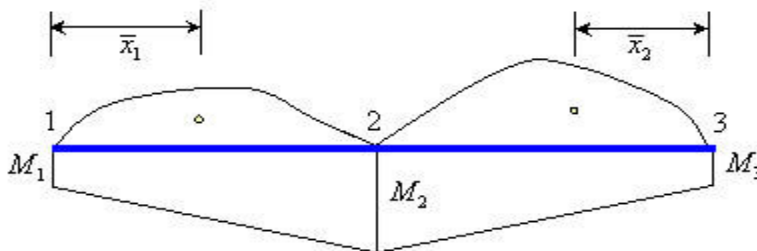


Figure 5.25(c)

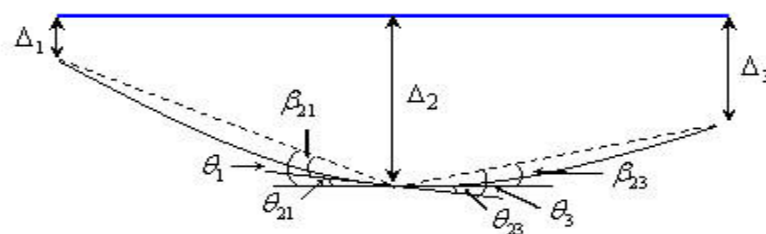


Figure 5.25(d)

The continuity of deflected shape of the beam at point 2 gives

$$\theta_{21} = \theta_{23} \quad \text{Equations 5.4}$$

From the Figure 5.25(d)

$$\theta_{21} = \theta_1 - \beta_{21} \text{ and } \theta_{23} = \theta_3 - \beta_{23}$$

Where

$$\theta_1 = (\Delta_1 - \Delta_2)/L_1 \text{ and } \theta_3 = (\Delta_3 - \Delta_2)/L_2$$

Using the bending moment diagrams shown in Figure 5.25(c) and the second moment area theorem,

$$\theta_{21} = [1/(L_1 * E * I_1)] * \{(M_1 L_1^2/6) + (M_2 L_1^2/3) + (A_1 X_1')\} \quad \text{Equations 5.7}$$

$$\theta_{23} = [1/(L_2 * E * I_2)] * \{(M_3 L_1^2/6) + (M_2 L_1^2/3) + (A_2 X_2')\} \quad \text{Equations 5.8}$$

Where A_1 and A_2 are the areas of the bending moment diagram of span 1-2 and 2-3, respectively considering the applied loading acting as simply supported beams.

Substituting from Equation (5.7) and Equation (5.8) in Equation (5.4) and Equation (5.5).

$$M_1(L_1/I_1) + 2M_2[(L_1/I_1) + (L_2/I_2)] + M_3(L_2/I_2) = -6 * A_1 * X_1' / (I_1 * L_1) - \\ 6 * A_2 * X_2' / (I_2 * L_2) + 6 * E [(\Delta_2 - \Delta_1)/L_1 + ((\Delta_2 - \Delta_3)/L_2)]$$

The above is known as *three moment equation*.

Sign Conventions

The M_1 , M_2 and M_3 are positive for sagging moment and negative for hogging moment. Similarly, areas A_1 , A_2 and A_3 are positive if it is sagging moment and negative for hogging moment. The displacements Δ_1 , Δ_2 and Δ_3 are positive if measured downward from the reference axis.

MOMENT DISTRIBUTION METHOD

Introduction

In the previous lesson we discussed the slope-deflection method. In slope-deflection analysis, the unknown displacements (rotations and translations) are related to the applied loading on the structure. The slope-deflection method results in a set of simultaneous equations of unknown displacements. The number of simultaneous equations will be equal to the number of unknowns to be evaluated. Thus one needs to solve these simultaneous equations to obtain displacements and beam end moments. Today, simultaneous equations could be solved very easily using a computer. Before the advent of electronic computing, this really posed a problem as the number of equations in the case of multistory building is quite large. The moment-distribution method proposed by Hardy Cross in 1932, actually solves these equations by the method of successive approximations. In this method, the results may be obtained to any desired degree of accuracy. Until recently, the moment-distribution method was very popular among

engineers. It is very simple and is being used even today for preliminary analysis of small structures. It is still being taught in the classroom for the simplicity and physical insight it gives to the analyst even though stiffness method is being used more and more. Had the computers not emerged on the scene, the moment-distribution method could have turned out to be a very popular method. In this lesson, first moment-distribution method is developed for continuous beams with unyielding supports.

Basic Concepts

In moment-distribution method, counterclockwise beam end moments are taken as positive. The counterclockwise beam end moments produce clockwise moments on the joint. Consider a continuous beam $ABCD$ as shown in Fig.18.1a. In this beam, ends A and D are fixed and hence, $\vartheta_A = \vartheta_D = 0$. Thus, the deformation of this beam is completely defined by rotations ϑ_B and C respectively. The required equation to evaluate θ_B and considering equilibrium of joints B and C . Hence,

$$\sum M_B = 0 \Rightarrow M_{BA} + M_{BC} = 0 \quad (18.1a)$$

$$\sum M_C = 0 \Rightarrow M_{CB} + M_{CD} = 0 \quad (18.1b)$$

According to slope-deflection equation, the beam end moments are written as

$$M_{BA} = M_{BA}^F + (2EI/L) (2\theta_B + \theta_A)$$

$(4EI/L)$ is known as stiffness factor for the beam AB and it is denoted by k_{AB} . M_{BA}^F is the fixed end moment at joint B of beam AB when joint B is fixed.

Thus,

$$M_{BA} = M_{BA}^F + K_{AB}\vartheta_B$$

$$M_{CD} = M_{CD}^F + K_{CD}\vartheta_C \quad (18.2)$$

In Fig.18.1b, the counterclockwise beam-end moments M_{BA} and M_{BC} produce

A clockwise moment M_B on the joint as shown in Fig.18.1b. To start with, in moment-distribution method, it is assumed that joints are locked i.e. joints are prevented from rotating. In such a case (vide Fig.18.1b),

$\vartheta_B = \vartheta_C = 0$, and hence

$$\begin{aligned} M_{BA} &= M_{BA}^F \\ M_{BC} &= M_{BC}^F \\ M_{CB} &= M_{CB}^F \\ M_{CD} &= M_{CD}^F \end{aligned} \quad (18.3)$$

Since joints B and C are artificially held locked, the resultant moment at joints B and C will not be equal to zero. This moment is denoted by M_B and is known as the unbalanced moment.

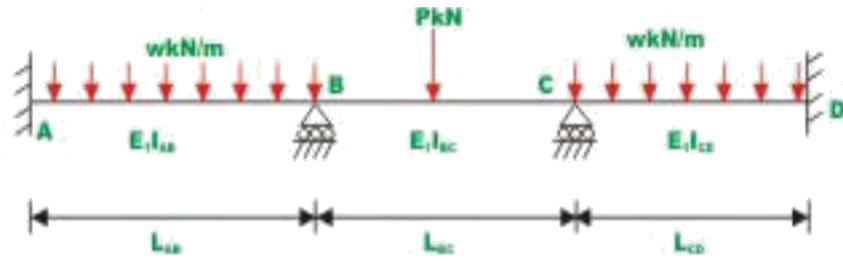


Fig. 18.1a Continuous Beam



Fig. 18.1b Continuous beam with fixed joints.

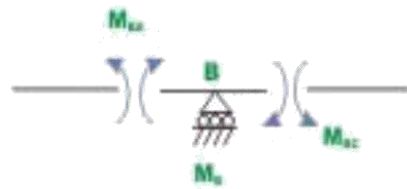


Fig. 18.1c Free - body diagram of joints B

Thus,

$$M_B = M_{BA}^F + M_{BC}^F$$

In reality joints are not locked. Joints B and C do rotate under external loads. When the joint B is unlocked, it will rotate under the action of unbalanced moment M_B . Let the joint B rotate by an angle θ_{B1} , under the action of M_B . This will deform the structure as shown in Fig.18.1d and introduces distributed moment M_{BA}^d, M_{BC}^d in the span BA and BC respectively as shown in the figure.

The unknown distributed moments are assumed to be positive and hence act in counterclockwise direction. The unbalanced moment is the algebraic sum of the fixed end moments and act on the joint in the clockwise direction. The unbalanced moment restores the equilibrium of the joint B . Thus,

$$\sum M_B = 0, M_{BA}^d + M_{BC}^d + M_B = 0 \quad (2.4)$$

The distributed moments are related to the rotation θ_{B1} by the slope- deflection equation.

$$M_{BA}^d = K_{BA} \theta_{B1}$$

$$M_{BC}^d = K_{BC} \theta_{B1} \quad (2.5)$$

Substituting equation (18.5) in (18.4), yields

$$\theta_{B1} (K_{BA} + K_{BC}) = -M_B$$

$$\theta_{B1} = -M_B / (K_{BA} + K_{BC})$$

In general, where summation is taken over all the members meeting at that particular joint. Substituting the value of θ_{B1} in equation (2.5), distributed moments are calculated. Thus, the ratio $\sum K_{BA} / \sum K$ is known as the distribution factor and is represented by DF_{BA} .

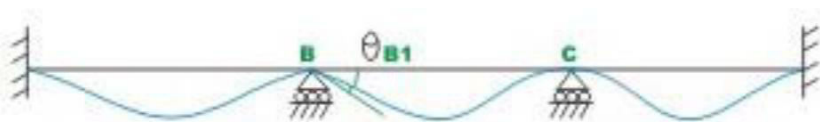
$$\text{Thus, } M_{BA}^d = -DF_{BA} \cdot M_B$$

$$M_{BC}^d = -DF_{BC} \cdot M_B \quad (2.8)$$

The distribution moments developed in a member meeting at B , when the joint B is unlocked and allowed to rotate under the action of unbalanced moment M_B is equal to a distribution factor times the unbalanced moment with its sign reversed.

As the joint B rotates under the action of the unbalanced moment, beam end moments are developed at ends of members meeting at that joint and are known as distributed moments. As the joint B rotates, it bends the beam and beam end moments at the far ends (i.e. at A and C) are developed. They are known as carry over moments. Now consider the beam BC of continuous beam $ABCD$.

When the joint B is unlocked, joint C is locked. The joint B rotates by θ_{B1} under the action of unbalanced moment M_B (vide Fig. 18.1e). Now from slope-deflection equations



The carry over moment is one half of the distributed moment and has the same sign. With the above discussion, we are in a position to apply moment-distribution method to statically indeterminate beam. Few problems are solved here to illustrate the procedure. Carefully go through the first problem, wherein the moment-distribution method is explained in detail.

Example

A continuous prismatic beam ABC (see Fig.2.2a) of constant moment of inertia is carrying a uniformly distributed load of 2 kN/m in addition to a concentrated load of 10 kN . Draw bending moment diagram. Assume that supports are unyielding

Solution

Assuming that supports B and C are locked, calculate fixed end moments developed in the beam due to externally applied load. Note that counterclockwise moments are taken as positive.

$$M_{AB} = 1.5 \text{ kN M.}$$

$$M_{BA} = -1.5 \text{ kN M.}$$

$$M_{BC} = 5 \text{ kN M.}$$

$$M_{CB} = -5 \text{ kN M.}$$

Before we start analyzing the beam by moment-distribution method, it is required to calculate stiffness and distribution factors.

$$\text{At } C: \sum K = EI$$

$$DF_{CB} = 1.0$$

Note that distribution factor is dimensionless. The sum of distribution factor at a joint, except when it is fixed is always equal to one. The distribution moments are developed only when the joints rotate under the action of unbalanced moment. In the case of fixed joint, it does not rotate and hence no distribution moments are developed and consequently distribution factor is equal to zero.

In Fig.18.2b the fixed end moments and distribution factors are shown on a working diagram. In this diagram B and C are assumed to be locked.

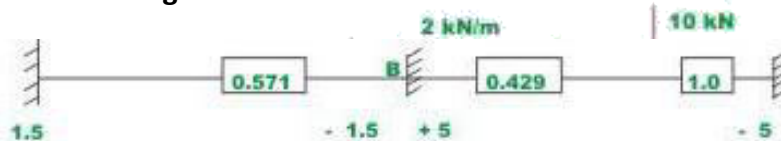


Fig. 18.2b

Now unlock the joint C . Note that joint C starts rotating under the unbalanced moment of 5 kN.m (counterclockwise) till a moment of -5 kN.m is developed (clockwise) at the joint.

This in turn develops a beam end moment of +5 kN.m (M_{CB}). This is the distributed moment and thus restores equilibrium. Now joint C is relocked and a line is drawn below +5 kN.m to indicate equilibrium. When joint C rotates, a carryover moment of +2.5 kN.m is developed at the B end of member BC. These are shown in Fig.18.2c.

When joint B is unlocked, it will rotate under an unbalanced moment equal to algebraic sum of the fixed end moments(+5.0 and -1.5 kN.m) and a carryover moment of +2.5 kN.m till distributed moments are developed to restore equilibrium. The unbalanced moment is 6 kN.m. Now the distributed moments M_{BC} and M_{BA} are obtained by multiplying the unbalanced moment with the corresponding distribution factors and reversing the sign. Thus, $M_{BC} = -2.574$ kN.m and $M_{BA} = -3.426$ kN.m. These distributed moments restore the equilibrium of joint B. Lock the joint B. This is shown in Fig.18.2d along with the carry over moments.

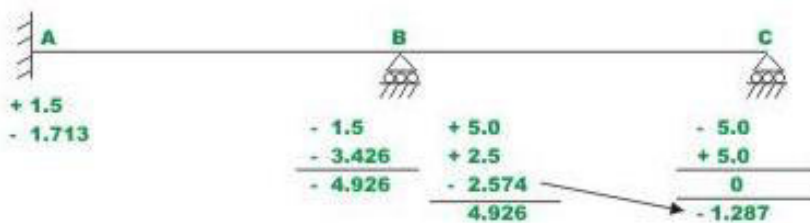


Fig. 18.2d

Now, it is seen that joint B is balanced. However joint C is not balanced due to the carry over moment -1.287 kN.m that is developed when the joint B is allowed to rotate. The whole procedure of locking and unlocking the joints C and B successively has to be continued till both joints B and C are balanced simultaneously. The complete procedure is shown in Fig.18.2e.

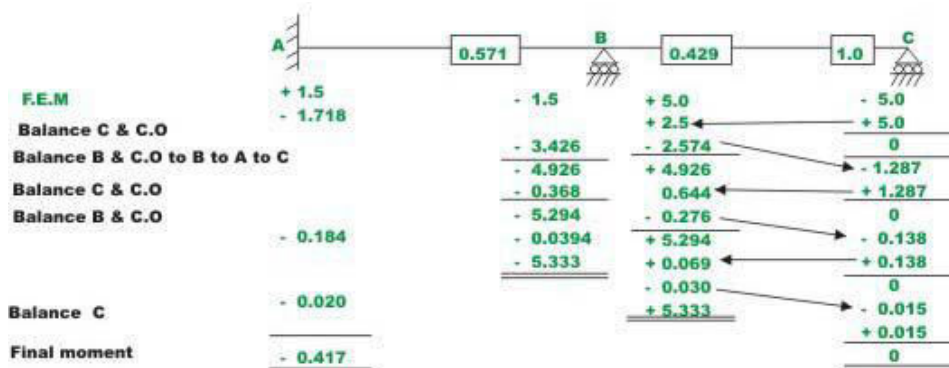


Fig. 18.2e Moment - distribution method : Computation

The iteration procedure is terminated when the change in beam end moments is less than say 1%. In the above problem the convergence may be improved if we leave the hinged end C unlocked after the first cycle. This will be discussed in the next section. In such a case the stiffness of beam BC gets modified. The above calculations can also be done conveniently in a tabular form as shown in Table 18.1. However the above working method is preferred in this course.

Table 18.1 Moment-distribution for continuous beam ABC

Joint	A	B		C
Member	AB	BA	BC	CB
Stiffness	1.333EI	1.333EI	EI	EI
Distribution factor		0.571	0.429	1.0
FEM kN.m	+1.5	-1.5	+5.0	-5.0
Balance joints C and C.O.	-1.713	-3.426	+2.5 -2.579	+5.0 0
		-4.926	+4.926	-1.287
Balance and C.O.			+0.644	1.287
Bala		-0.368	-0.276	-0.13

nce and C.O.				8
Balance C	-0.184	-5.294	+5.294	0.138
C.O.			+0.069	0
Bala nce and C.O.	-0.02	-0.039	-0.030	-0.015
Balance C				+0.015
Bala nced momen ts in kN. m	-0.417	-5.333	+5.333	0

Example

Draw the bending moment diagram for the continuous beam *ABCD* loaded as shown in Fig.18.4a. The relative moment of inertia of each span of the beam is also shown in the figure.

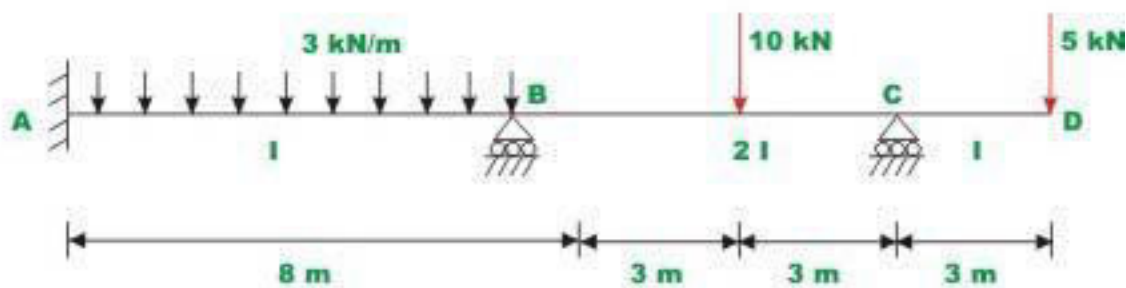


Fig. 18.4a Example 18.3

Solution

Note that joint C is hinged and hence stiffness factor BC gets modified. Assuming that the supports are locked, calculate fixed end moments. They are

$$M_{AB}^F = 16 \text{ kN.m}$$

$$M_{BA}^F = -16 \text{ kN.m}$$

$$M_{BC}^F = 7.5 \text{ kN.m}$$

$$M_{CB}^F = -7.5 \text{ kN.m, and}$$

$$M_{CD}^F = 15 \text{ kN.m}$$

In the next step calculate stiffness and distribution factors

Now all the calculations are shown in Fig.18.4b

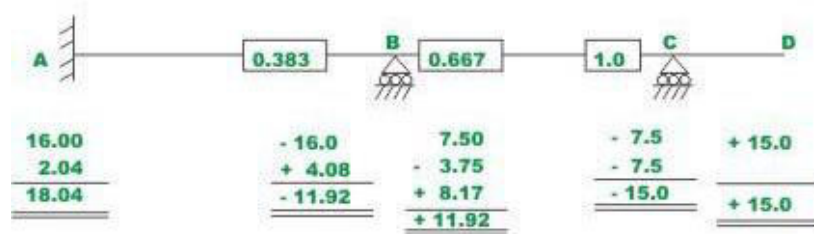


Fig. 18.4b Computation

This problem has also been solved by slope-deflection method (see example 14.2).The bending moment diagram is shown in Fig.18.4c.

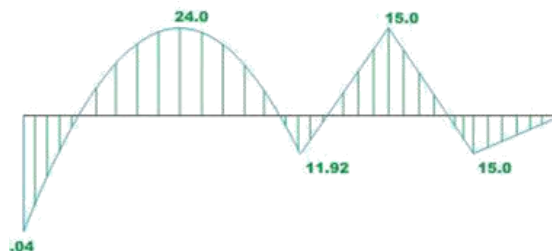


Fig. 18.4c Bending - moment diagram

Summary

An introduction to the moment-distribution method is given here. The moment-distribution method actually solves these equations by the method of successive approximations. Various terms such as stiffness factor, distribution factor, unbalanced moment, distributing moment and carry-over-moment are defined in this lesson.

Unit III

Indeterminate Structures - II: Analysis of beams and frames by slope Deflection method, Column Analogy method.

Slope deflection Method

Introduction

As pointed out earlier, there are two distinct methods of analysis for statically indeterminate structures depending on how equations of equilibrium, load displacement and compatibility conditions are satisfied: 1) force method of analysis and (2) displacement method of analysis. In the last module, force method of analysis was discussed. In this module, the displacement method of analysis will be discussed. In the force method of analysis, primary unknowns are forces and compatibility of displacements is written in terms of pre-selected redundant reactions and flexibility coefficients using force displacement relations. Solving these equations, the unknown redundant reactions are evaluated. The remaining reactions are obtained from equations of equilibrium.

As the name itself suggests, in the displacement method of analysis, the primary unknowns are displacements. Once the structural model is defined for the problem, the unknowns are automatically chosen unlike the force method. Hence this method is more suitable for computer implementation. In the displacement method of analysis, first equilibrium equations are satisfied. The equilibrium of forces is written by expressing the unknown joint displacements in terms of load by using load displacement relations. These equilibrium equations are solved for unknown joint displacements. In the next step, the unknown reactions are computed from compatibility equations using force displacement relations. In displacement method, three methods which are closely related to each other will be discussed.

1. Slope-Deflection Method
2. Moment Distribution Method
3. Direct Stiffness Method

In this module first two methods are discussed and direct stiffness method is treated in the next module. All displacement methods follow the above general procedure. The Slope-deflection and moment distribution methods were extensively used for many years before the compute era. After the revolution occurred in the field of computing only direct stiffness method is preferred.

Degrees of freedom

In the displacement method of analysis, primary unknowns are joint displacements which are commonly referred to as the degrees of freedom of the structure. It is necessary to consider all the independent degrees of freedom while writing the equilibrium equations. These degrees of freedom are specified at supports, joints and at the free ends. For example, a propped cantilever beam (see Fig.14.01a) under the action of load P will

undergo only rotation at B if axial deformation is neglected. In this case kinematic degree of freedom of the beam is only one i.e. ϑ_B as shown in the figure.

In Fig 14.01 (b), we have nodes at A, B, C and D. Under the action of lateral loads, P_1 , P_2 and P_3 , this continuous beam deform as shown in the figure. Here axial deformations are neglected. For this beam we have five degrees of freedom ϑ_A , ϑ_B , ϑ_C , ϑ_D and Δ as indicated in the figure. In Fig.14.02a, a symmetrical plane frame is loaded symmetrically. In this case we have only two degrees of freedom ϑ_B and ϑ_C . Now consider a frame as shown in Fig.14.02b. It has three degrees of freedom viz. ϑ_B , ϑ_C and Δ as shown. Under the action of horizontal and vertical load, the frame will be displaced as shown in the figure. It is observed that nodes at B and C undergo rotation and also get displaced horizontally by an equal amount.

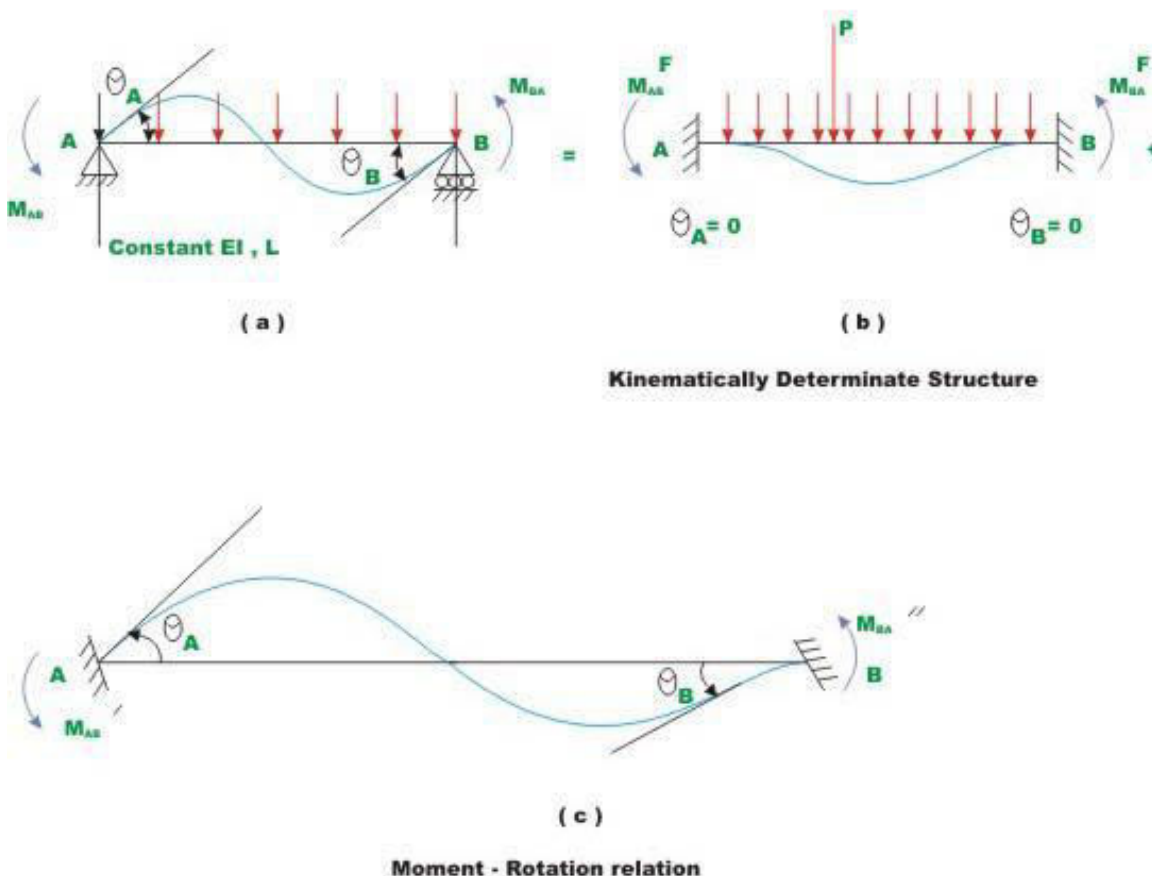


Figure 14.01: Derivation of slope – deflection equation.

Hence in plane structures, each node can have at the most one linear displacement and one rotation. In this module first slope-deflection equations as applied to beams and rigid frames will be discussed.

Slope-Deflection Equations

Consider a typical span of a continuous beam AB as shown in Fig.14.1. The beam has constant flexural rigidity EI and is subjected to uniformly distributed loading and concentrated loads as shown in the figure. The beam is kinematically indeterminate to

second degree. In this lesson, the slope-deflection equations are derived for the simplest case i.e. for the case of continuous beams with unyielding supports. In the next lesson, the support settlements are included in the slope-deflection equations.

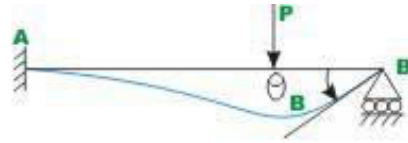


Fig. 14.01

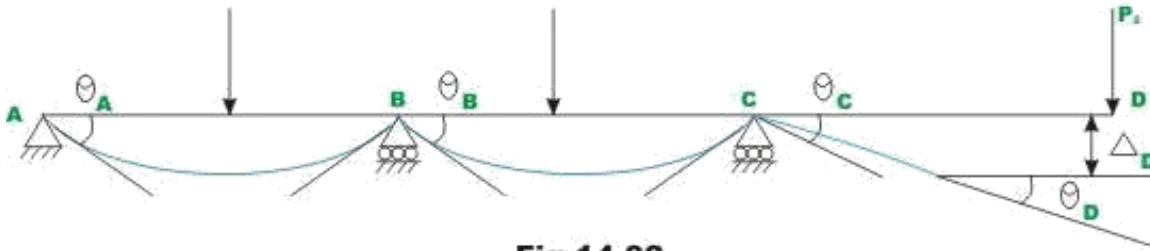
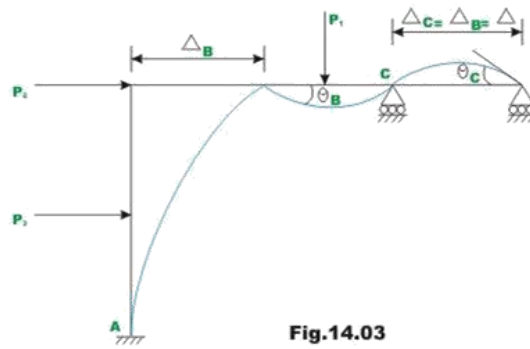


Fig.14.02

For this problem, it is required to derive relation between the joint end moments M_{AB} and M_{BA} in terms of joint rotations ϑ_A and ϑ_B and loads acting on the beam. Two subscripts are used to denote end moments. For example, end moments M_{AB} denote moment acting at joint A of the member AB. Rotations of the tangent to the elastic curve are denoted by one subscript. Thus, ϑ_A denotes the rotation of the tangent to the elastic curve at A. The following sign conventions are used in the slope-deflection equations (1) Moments acting at the ends of the member in counterclockwise direction are taken to be positive. (2) The rotation of the tangent to the elastic curve is taken to be positive when the tangent to the elastic curve has rotated in the counterclockwise direction from its original direction. The slope-deflection equations are derived by superimposing the end moments developed due to (1) applied loads (2) rotation ϑ_A (3)

Rotation ϑ_B . This is shown in Fig.14.2 (a)-(c). In Fig. 14.2(b) a kinematically determinate structure is obtained. This condition is obtained by modifying the support conditions to fixed so that the unknown joint rotations become zero. The structure shown in Fig.14.2 (b) is known as kinematically determinate structure or restrained structure. For this case, the end moments are denoted by M_{AB}^F and M_{BA}^F . The fixed end moments are evaluated by force-method of analysis as discussed in the previous module. For example for fixed- fixed beam subjected to uniformly distributed load, the fixed-end moments are shown in Fig.14.3.



The fixed end moments are required for various load cases. For ease of calculations, fixed end forces for various load cases are given at the end of this lesson. In the actual structure end A rotates by ϑ_A and end B rotates by ϑ_B . Now it is required to derive a relation relating ϑ_A and ϑ_B with the end moments M'_{AB} and M'_{BA} . Towards this end, now consider a simply supported beam acted by moment M'_{AB} at A as shown in Fig. 14.4. The end moment M'_{AB} deflects the beam as shown in the figure. The rotations ϑ'_A and ϑ'_B are calculated from moment-area theorem.

$$\Theta'_A = (M'_{AB} * L) / (3EI) \quad 3.1a$$

$$\Theta'_B = (M'_{AB} * L) / (6EI) \quad 3.1b$$

Now a similar relation may be derived if only M'_{BA} is acting at end B (see Fig. 14.4).

$$\Theta''_B = (M'_{BA} * L) / (3EI) \quad 3.2a$$

$$\Theta''_A = (M'_{BA} * L) / (6EI) \quad 3.2b$$

Now combining these two relations, we could relate end moments acting at A and B to rotations produced at A and B as (see Fig. 14.2c)

$$\Theta_A = (M'_{AB} * L) / (3EI) - (M'_{BA} * L) / (6EI) \quad 3.3a$$

$$\Theta_B = (M'_{BA} * L) / (3EI) - (M'_{AB} * L) / (6EI) \quad 3.3b$$

Solving for M_{AB} and M_{BA} in terms of θ_A and θ_B ,

$$M'_{AB} = (2EI/L) (2 \theta_A + \theta_B) \quad 3.4$$

$$M'_{BA} = (2EI/L) (2 \theta_B + \theta_A) \quad 3.5$$

Now writing the equilibrium equation for joint moment at A (see Fig. 14.2).

$$M_{AB} = M^F_{AB} + M'_{AB} \quad 3.6a$$

Similarly writing equilibrium equation for joint B

$$M_{BA} = M^F_{BA} + M'_{BA} \quad 3.6b$$

Substituting the values of M'_{AB} and M'_{BA}

$$M_{AB} = M^F_{AB} + (2EI/L) (2 \theta_A + \theta_B) \quad 3.7a$$

$$M_{BA} = M_{BA}^F + \frac{2EI}{L} (2\theta_B + \theta_A) \quad 3.7b$$

Sometimes one end is referred to as near end and the other end as the far end. In that case, the above equation may be stated as the internal moment at the near end of the span is equal to the fixed end moment at the near end due to external loads plus $\frac{2EI}{L}$ times the sum of twice the slope at the near end and the slope at the far end. The above two equations (14.7a) and (14.7b) simply referred to as slope-deflection equations. The slope-deflection equation is nothing but a load displacement relationship.

3.3 Application of Slope-Deflection Equations to Statically Indeterminate Beams

The procedure is the same whether it is applied to beams or frames. It may be summarized as follows:

- 1) Identify all kinematic degrees of freedom for the given problem. This can be done by drawing the deflection shape of the structure. All degrees of freedom are treated as unknowns in slope-deflection method.
- 2) Determine the fixed end moments at each end of the span to applied load. The table given at the end of this lesson may be used for this purpose.
- 3) Express all internal end moments in terms of fixed end moments and near end, and far end joint rotations by slope-deflection equations.
- 4) Write down one equilibrium equation for each unknown joint rotation. For example, at a support in a continuous beam, the sum of all moments corresponding to an unknown joint rotation at that support must be zero.
- 5) Write down as many equilibrium equations as there are unknown joint rotations.
- 6) Solve the above set of equilibrium equations for joint rotations.
- 7) Now substituting these joint rotations in the slope-deflection equations evaluate the end moments.
- 8) Determine all rotations.

Example

A continuous beam ABC is carrying uniformly distributed load of 2 kN/m in addition to a concentrated load of 20 kN as shown in Fig.14.5a. Draw bending moment and shear force diagrams. Assume EI to be constant.

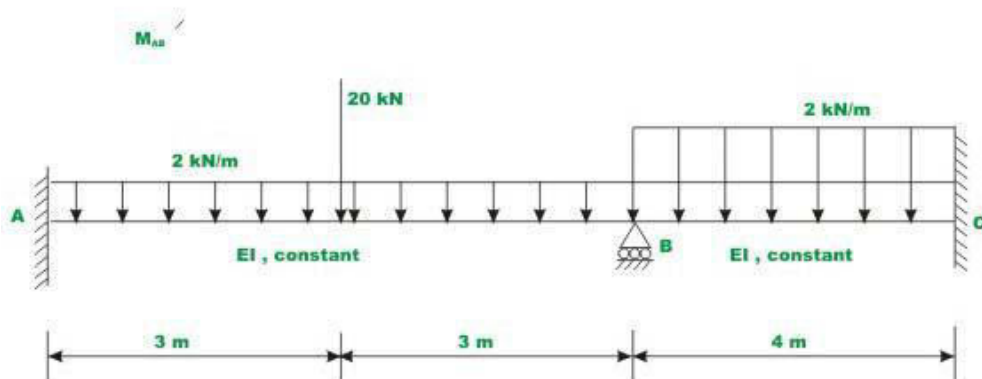


Fig. 14.5(a) Example 14.1

(a) Degrees of freedom

It is observed that the continuous beam is kinematically indeterminate to first degree as only one joint rotation θ_B is unknown. The deflected shape /elastic curve of the beam is drawn in Fig.14.5b in order to identify degrees of freedom.

By fixing the support or restraining the support B against rotation, the fixed-fixed beams are obtained as shown in Fig. C

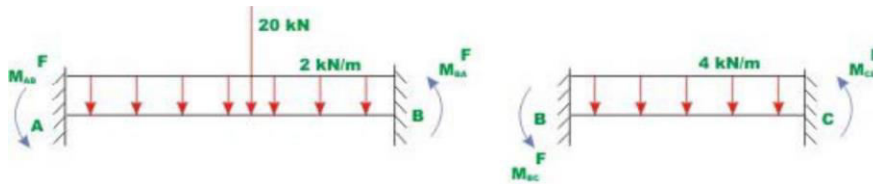


Fig. 14.5 (c) Restrained Structure.

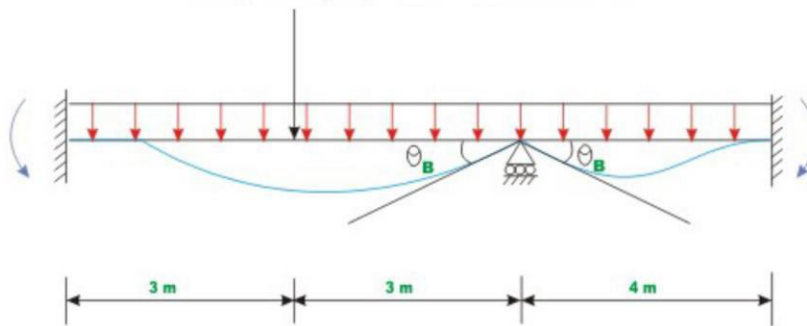


Fig. 14.5 (b) Elastic curve of the beam with unknown displacement component θ_B

(b). Fixed end moments M_{AB}^F , M_{BA}^F , M_{BC}^F and M_{CB}^F are calculated referring to the Fig. 14 and following the sign conventions that counterclockwise moments are positive.

$$M_{AB}^F = (2 \cdot 6 \cdot 6)/12 + (20 \cdot 3 \cdot 3 \cdot 3)/(6 \cdot 6) = 21 \text{ kN M.}$$

$$M_{BA}^F = - 21 \text{ kN M.}$$

$$M_{BC}^F = (4 \cdot 4)/12 = 5.33 \text{ kN M.}$$

$$M_{CB}^F = - 5.33 \text{ kN M.}$$

(c) Slope-deflection equations

Since ends A and C are fixed, the rotation at the fixed supports is zero, $\theta_A = \theta_C = 0$. Only one non-zero rotation is to be evaluated for this problem. Now, write slope-deflection equations for span AB and BC .

$$M_{AB} = M_{AB}^F + (2EI/L) (2\theta_A + \theta_B) = 21 + (2EI/6) \theta_B$$

$$M_{BA} = M_{BA}^F + (2EI/L) (2\theta_B + \theta_A) = -21 + (4EI/6) \theta_B$$

$$M_{BC} = 5.33 + EI\theta_B$$

$$M_{CB} = -5.33 + 0.5EI\theta_B$$

(d) Equilibrium equations

In the above four equations (2-5), the member end moments are expressed in terms of unknown rotation θ_B . Now, the required equation to solve for the rotation θ_B is the moment equilibrium equation at support B . The free body diagram of support B along with the support moments acting on it is shown in Fig. 14.5d. For, moment equilibrium at support B , one must have,

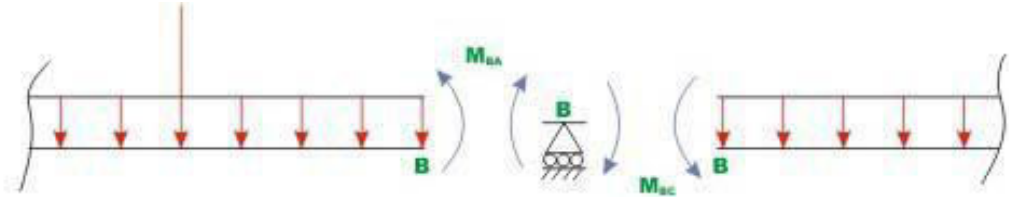


Fig. 14.5 d Free- body diagram of the joint B

$$\Sigma M_B = 0 \quad M_{BA} + M_{BC} = 0$$

Substituting the values of M_{BA} and M_{BC} in the above equilibrium equation,

$$-21 + (4EI/6) \theta_B + 5.33 + EI\theta_B = 0$$

$$\theta_B = 9.4/(EI)$$

(e) End moments

After evaluating θ_B , substitute it in equations (2-5) to evaluate beam end moments. Thus,

$$M_{AB} = 24.133 \text{ kN M.}$$

$$M_{BA} = -14.733 \text{ kN M.}$$

$$M_{BC} = 14.733 \text{ kN M.}$$

$$M_{CB} = -0.63 \text{ kN M.}$$

(f) Reactions

Now, reactions at supports are evaluated using equilibrium equations (vide Fig. 14.5e)

$$R_A \times 6 + 14.733 - 20 \times 3 - 2 \times 6 \times 3 - 24.133 = 0$$

$$R_A = 17.567 \text{ kN}(\uparrow)$$

$$R_{BL} = 20 + 12 - 17.567 = 14.433 \text{ kN}(\uparrow)$$

$$R_{BR} \times 4 - 14.733 - (4 \times 4 \times 2) + 0.63 = 0$$

$$R_{BR} = 11.526 \text{ kN}(\uparrow)$$

$$R_C = 16 - 11.526 = 4.47 \text{ kN}(\uparrow)$$

The shear force and bending moment diagrams are shown in Fig. 14.5f.

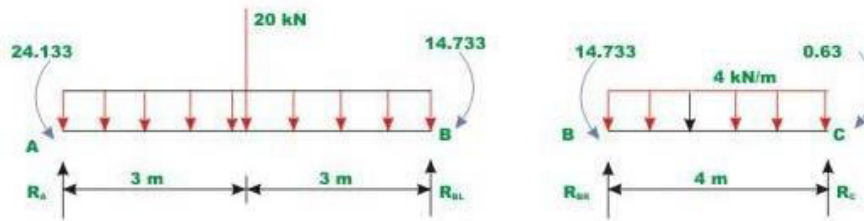
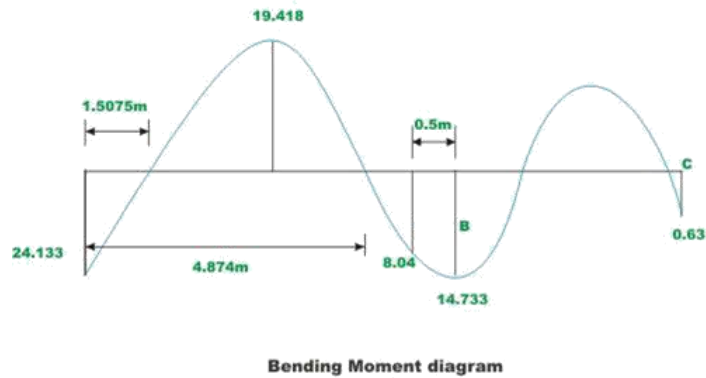
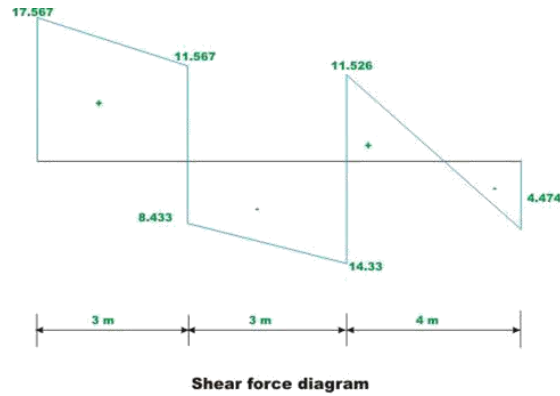
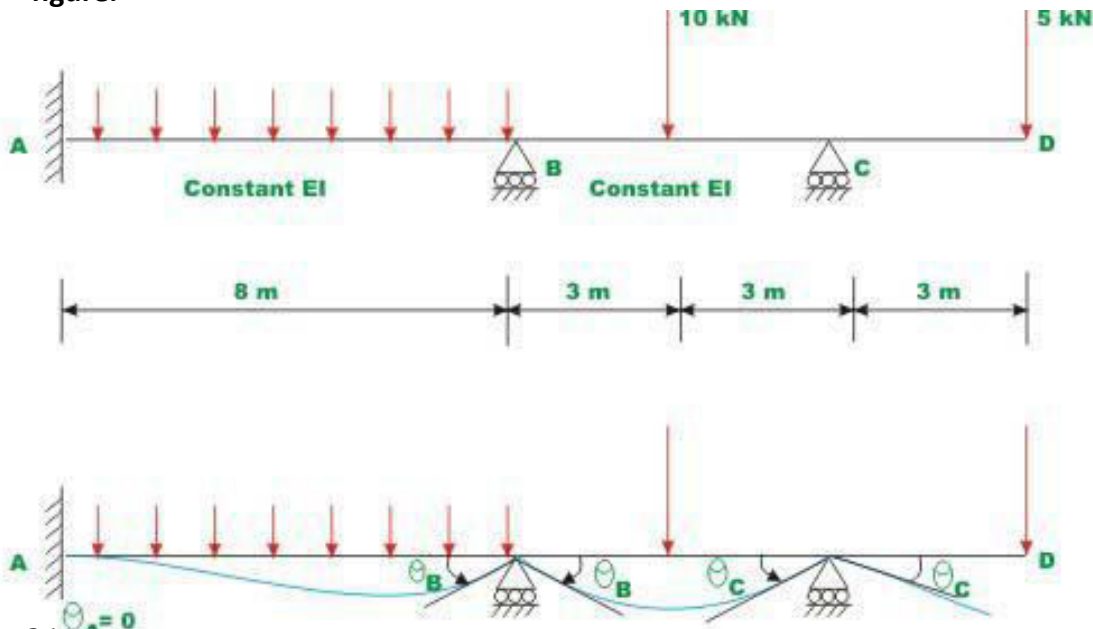


Fig. 14.5 (e) Free - body diagram of two members



Example

Draw shear force and bending moment diagram for the continuous beam *ABCD* loaded as shown in Fig.14.6a. The relative stiffness of each span of the beam is also shown in the figure.



For the cantilever beam portion CD , no slope-deflection equation need to be written as there is no internal moment at end D . First, fixing the supports at B and C , calculate the fixed end moments for span AB and BC . Thus,

$$M_{AB}^F = (3 \cdot 8 \cdot 8)/12 = 16 \text{ kN M.}$$

$$M_{BA}^F = - 16 \text{ kN M.}$$

$$M_{BC}^F = (10 \cdot 3 \cdot 3)(6 \cdot 6) = 7.5 \text{ kN M.}$$

$$M_{CB}^F = - 7.5 \text{ kN M.}$$

In the next step write slope-deflection equation. There are two equations for each span of the continuous beam.

$$M_{AB} = 16 + (2EI/L) \theta_B = 16 + 0.25EI \theta_B$$

$$M_{BA} = - 16 + 0.5EI \theta_B$$

$$M_{BC} = 7.5 + (2EI/L) (2\theta_B + \theta_C) = 7.5 + 1.334EI \theta_B + 0.667EI \theta_C$$

$$M_{CB} = - 7.5 + 1.334EI \theta_C + 0.667EI \theta_B$$

Equilibrium equations

The free body diagram of members AB , BC and joints B and C are shown in Fig.14.6b. One could write one equilibrium equation for each joint B and C .



Fig. 14.6 b Free - body diagrams of joints B and C along with members

Support B,

$$\Sigma M_B = 0 \quad M_{BA} + M_{BC} = 0$$

$$\Sigma M_C = 0 \quad M_{BC} + M_{CD} = 0$$

We know that $M_{CD} = 15 \text{ kN.M}$

$M_{CB} = 15 \text{ kN.M}$. Substituting the values of M_{CB} and M_{CD} in the above equation we get

$$\theta_B = 8.164 \text{ and } \theta_C = 9.704$$

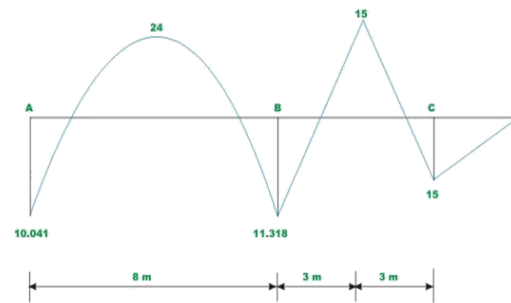
Substituting θ_B θ_C in the slope-deflection equations, we get

$$M_{Ab} = 18.04 \text{ kN M.}$$

$$M_{Ba} = 11.918 \text{ kN M.}$$

$$M_{BC} = - 11.918 \text{ kN M.}$$

$$M_{CB} = - 15.0 \text{ kN M.}$$



Reactions are obtained from equilibrium equations (ref. Fig. 14.6c)

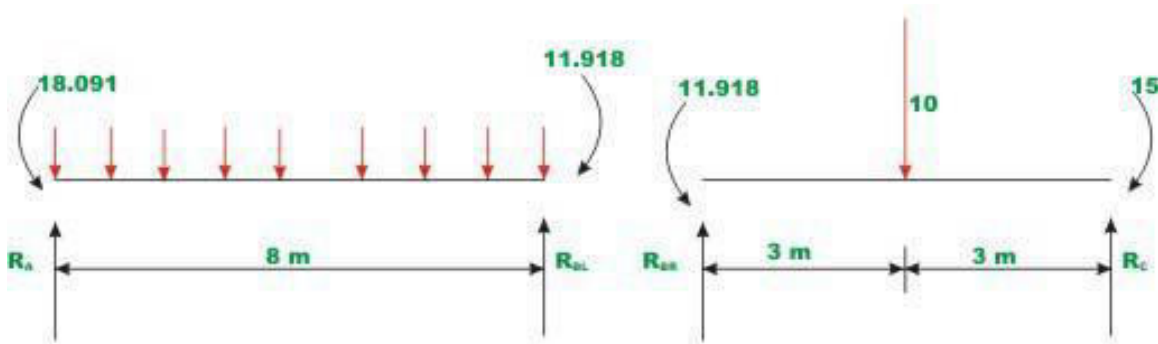


Fig. 14.6 c Computation of reactions

$$R_A \times 8 - 18.041 \times 4 + 11.918 = 0$$

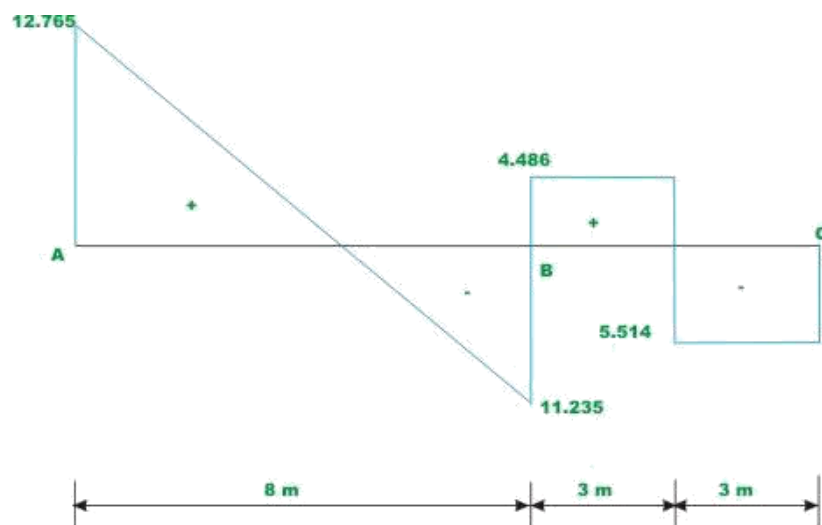
$$R_A = 12.765 \text{ kN}$$

$$R_{BR} = 5 - 0.514 \text{ kN} = 4.486 \text{ kN}$$

$$R_{BL} = 11.235 \text{ kN}$$

$$R_C = 5 + 0.514 \text{ kN} = 5.514 \text{ kN}$$

The shear force and bending moment diagrams are shown in Fig. 14.6d.



	$M_A = \frac{Pab^2}{L^2}$ $M_B = -\frac{Pab^2}{L^2}$
	$M_A = \frac{wL^2}{12}$ $M_B = -\frac{wL^2}{12}$
	$M_A = \frac{11wL^2}{192}$ $M_B = -\frac{5wL^2}{192}$
	$M_A = \frac{wL^2}{20}$ $M_B = -\frac{wL^2}{30}$
	$M_A = \frac{Mb}{L^2} (2a - b)$ $M_B = \frac{Ma}{L^2} (2b - a)$
	$M_A = \frac{P}{L^2} (b^2a + \frac{a^3b}{2})$ $M_B = 0$
	$M_A = \frac{wL^2}{8}$ $M_B = 0$

Fig. 14.7 Table of fixed end moments

For ease of calculations, fixed end forces

Summary

In this lesson the slope-deflection equations are derived for beams with unyielding supports. The kinematically indeterminate beams are analysed by slope-deflection equations. The advantages of displacement method of analysis over force method of analysis are clearly brought out here. A couple of examples are solved to illustrate the slope-deflection equations.

Column Analogy Method

Introduction

The column analogy method was also proposed by Prof. Hardy Cross and is a powerful technique to analyze the beams with fixed supports, fixed ended gable frames, closed frames & fixed arches etc., These members may be of uniform or variable moment of inertia throughout their lengths but the method is ideally suited to the calculation of the stiffness factor and the carryover factor for the members having variable moment of inertia. The method is strictly applicable to a maximum of 3rd degree of indeterminacy. This method is essentially an indirect application of the consistent deformation method.

The method is based on a mathematical similarity (i.e. analogy) between the stresses developed on a column section subjected to eccentric load and the moments imposed on a member due to fixity of its supports. In the analysis of actual engineering structures of modern times, so many analogies are used like slab analogy, and shell analogy etc. In all these methods, calculations are not made directly on the actual structure but, in fact it is always assumed that the actual structure has been replaced by its mathematical model and the calculations are made on the model. The final results are related to the actual structure through same logical engineering interpretation.

In the method of column analogy, the actual structure is considered under the action of applied loads and the redundant acting simultaneously on a BDS. The load on the top of the analogous column is usually the B.M.D. due to applied loads on simple spans and therefore the reaction to this applied load is the B.M.D. due to redundant on simple spans considers the following fixed ended loaded beam.

The resultant of B.M.D's due to applied loads does not fall on the midpoint of analogous column section which is eccentrically loaded.

M_s diagram = BDS moment diagram due to applied loads.

M_i diagram = Indeterminate moment diagram due to redundant.

If we plot (+ve) B.M.D. above the zero line and (–ve) B.M.D below the zero line (both on compression sides due to two sets of loads) then we can say that these diagrams have been plotted on the compression side. (The conditions from which M_A & M_B can be determined, when the method of consistent deformation is used, are as follows). From the Geometry requirements, we know that

(1) The change of slope between points A & B = 0; or sum of area of moment diagrams between A & B = 0 (note that EI = Constant), or area of moment diagrams of figure b = area of moment diagram of figure c.

(2) The deviation of point B from tangent at A = 0; or sum of moment of moment diagrams between A & B about B = 0, or Moment of moment diagram of figure(b) about B = moment of moment diagram of figure (c) about B. Above two requirements can be stated as follows.

(1) Total load on the top is equal to the total pressure at the bottom and;

(2) Moment of load about B is equal to the moment of pressure about B), indicates that the analogous column is on equilibrium under the action of applied loads and the redundant.

7.1. SIGN CONVENTIONS:– It is necessary to establish a sign convention regarding the nature of the applied load (M_s –diagram) and the pressures acting at the base of the analogous column (M_i –diagram.)

1. Load (P) on top of the analogous column is downward if M_s/EI diagram is (+ve) which means that it causes compression on the outside or (sagging) in BDS vice-versa. If EI is constant, it can be taken equal to units.

2. Upward pressure on bottom of the analogous column (M_i –diagram) is considered as (+ve).

3. Moment (M) at any point of the given indeterminate structure (maximum to 3rd degree) is given by the formula.

$M = M_s - M_i$, which is (+ve) if it causes compression on the outside of members

In the last lesson, slope-deflection equations were derived without considering the rotation of the beam axis. In this lesson, slope-deflection equations are derived considering the rotation of beam axis. In statically indeterminate structures, the beam axis rotates due to support yielding and this would in turn induce reactions and stresses in the structure. Hence, in this case the beam end moments are related to rotations, applied loads and beam axes rotation. After deriving the slope-deflection equation in section 15.2, few problems are solved to illustrate the procedure.

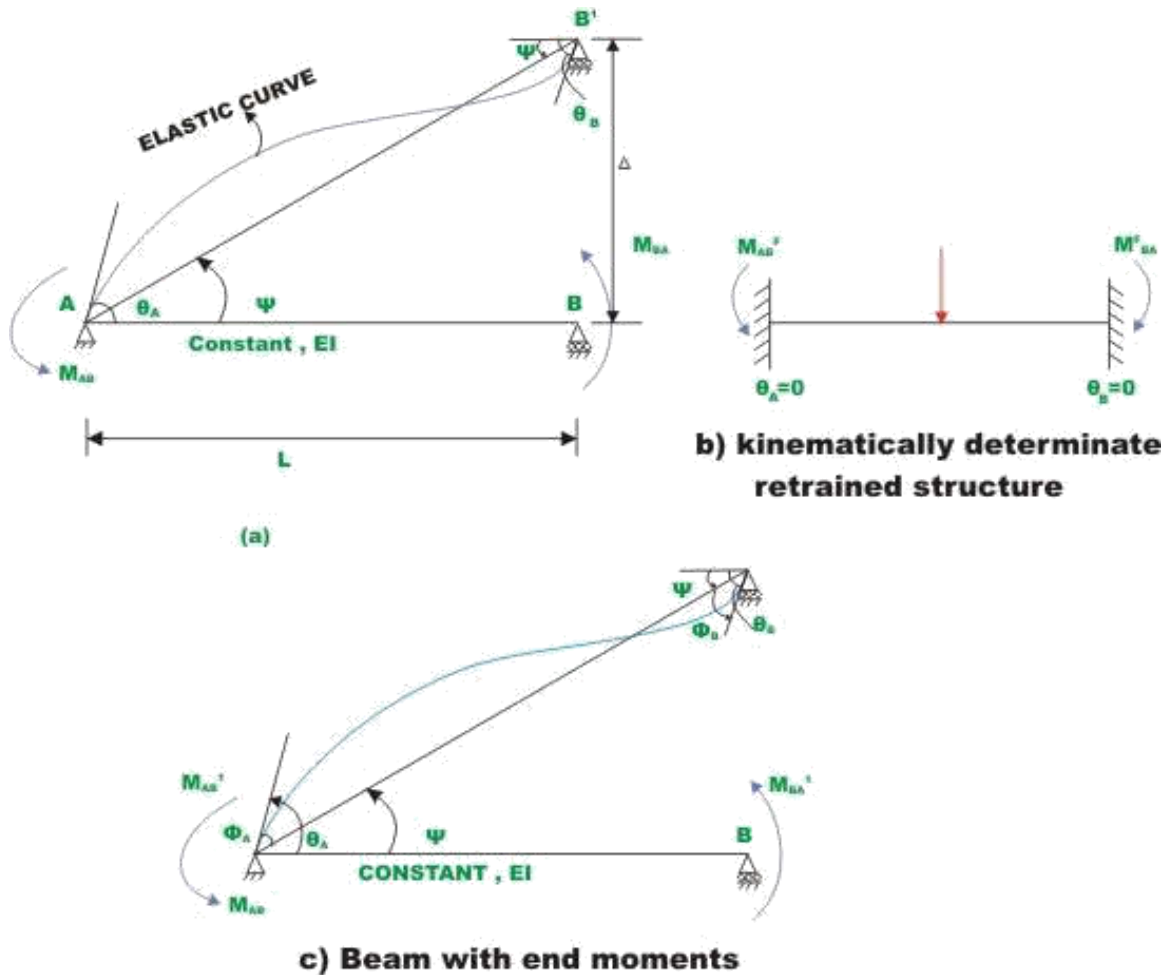


Figure 15.1

Now superposing the fixed end moments due to external load and end moments due to displacements, the end moments in the actual structure is obtained .Thus (see Fig.15.1)

In the above equations, it is important to adopt consistent sign convention. In the above derivation is taken to be negative for downward displacements. In the continuous beam ABC, two rotations ϑ_B and ϑ_C need to be evaluated.

Hence, beam is kinematically indeterminate to second degree. As there is no external load on the beam, the fixed end moments in the restrained beam are zero (see Fig.15.2b).

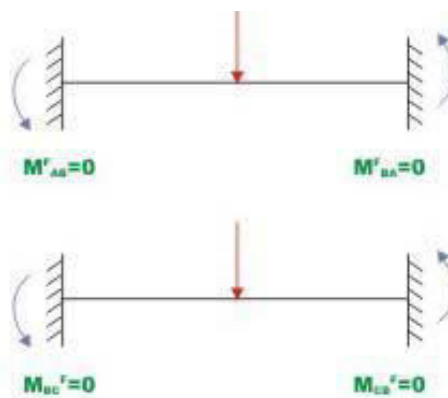


Figure 15.2 (b)

For each span, two slope-deflection equations need to be written. In span AB , B is below A . Hence, the chord AB rotates in clockwise direction. Thus, ψ_{AB} is taken as negative. In span BC , the support C is above support B , Hence the chord joining $B'C$ rotates in anticlockwise direction.

$$\psi_{BC} = \psi_{CB} = 1 \times 10^{-3}$$

Writing slope-deflection equations for span BC ,

$$M_{BC} = 0.8EI\vartheta_B + 0.4EI\vartheta_C - 1.2 \times 10^{-3} EI$$

$$M_{CB} = 0.8EI\vartheta_C + 0.4EI\vartheta_B - 1.2 \times 10^{-3} EI$$

Now, consider the joint equilibrium of support B (see Fig.15.2c)

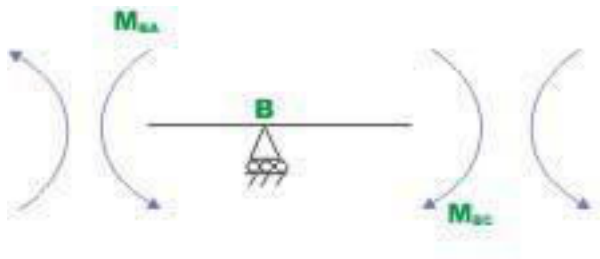


Fig 15.2c Free body diagram of joint B

$$M_{BA} + M_{BC} = 0$$

Substituting the values of M_{BA} and M_{BC} in equation (6),

$$0.8EI\vartheta_B + 1.2 \times 10^{-3} EI + 0.8EI\vartheta_B + 0.4EI\vartheta_C - 1.2 \times 10^{-3} EI = 0$$

$$\text{Simplifying, } 1.6\vartheta_B + 0.4\vartheta_C = 1.2 \times 10^{-3}$$

Also, the support C is simply supported and hence, $M_{CB} = 0$

$$M_{CB} = 0 = 0.8\vartheta_C + 0.4\vartheta_B - 1.2 \times 10^{-3} EI$$

$$0.8\vartheta_C + 0.4\vartheta_B = 1.2 \times 10^{-3}$$

We have two unknowns ϑ_B and ϑ_C and there are two equations in ϑ_B and ϑ_C . Solving equations (7) and (8)

$$= -0.4286 \times 10^{-3} \text{ radians}$$

$$= 1.7143 \times 10^{-3} \text{ radians} \quad (9)$$

Substituting the values of ϑ_B , ϑ_C and EI in slope-deflection equations,

$$M_{AB} = 82.285 \text{ kN.m} \quad M_{BA} = 68.570 \text{ kN.m}$$

$$M_{BC} = -68.573 \text{ kN.m} \quad M_{CB} = 0 \text{ kN.m}$$

Reactions are obtained from equations of static equilibrium (vide Fig.15.2d)

In beam AB,

$$\sum M_B = 0, R_A = 30.171 \text{ kN}(\uparrow)$$

$$R_{BL} = -30.171 \text{ kN}(\downarrow)$$

$$R_{BR} = -13.714 \text{ kN}(\downarrow)$$

$$R_C = 13.714 \text{ kN}(\uparrow)$$

The shear force and bending moment diagram is shown in Fig.15.2e and elastic curve is shown in Fig.15.2f.

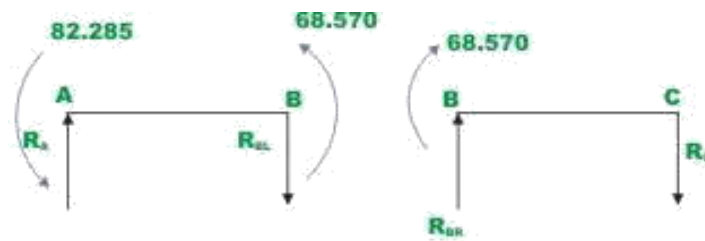


Fig 15.2d Computation of reactions

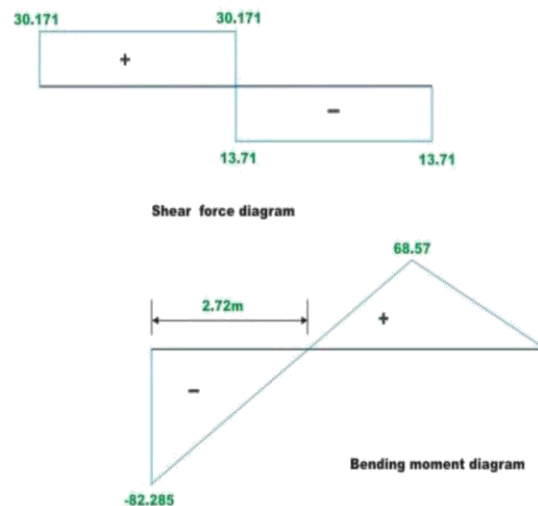
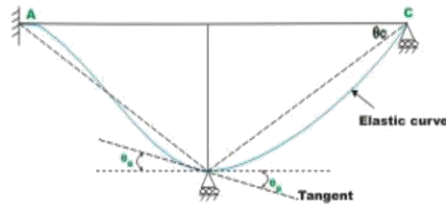


Figure 15.2e Shear force and bending moment diagram



15.2 f Elastic curve

Example

A continuous beam $ABCD$ is carrying a uniformly distributed load of 5 kN/m as shown in Fig.15.3a. Compute reactions and draw shear force and bending moment diagram due to following support settlements.

Support B 0.005m vertically downwards

Support C 0.01 m vertically downwards

Assume $E = 200 \text{ GPa}$, $I = 1.35 \times 10^{-3} \text{ m}^4$

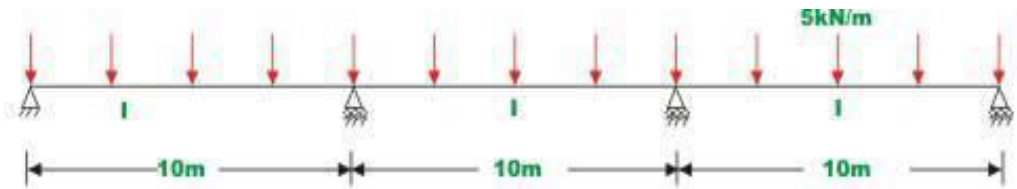


Fig 15.3a Continuous beam of Example 15.2

In the above continuous beam, four rotations ϑ_A , θ_B , θ_C and θ_D are to be evaluated. One equilibrium equation can be written at each support. Hence, solving the four equilibrium equations, the rotations are evaluated and hence the moments from slope-deflection equations. Now consider the kinematically restrained beam as shown in Fig.15.3b.

Referring to standard tables the fixed end moments may be evaluated. Otherwise one could obtain fixed end moments from force method of analysis. The fixed end moments in the present case are (vide fig.15.3b)

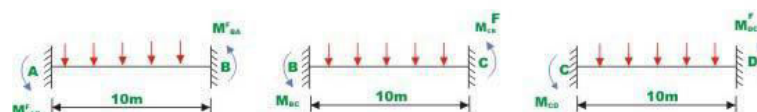


Fig 15.3b Kinematically restrained beam

$$M_{AB}^F = 41.667 \text{ kN.m}$$

$$M_{BA}^F = -41.667 \text{ kN.m (clockwise)}$$

$$M_{BC}^F = 41.667 \text{ kN.m (counterclockwise)}$$

$$M_{CB}^F = -41.667 \text{ kN.m (clockwise)}$$

$$M_{CD}^F = 41.667 \text{ kN.m (counter clockwise)}$$

$$M_{DC}^F = -41.667 \text{ kN.m (clockwise)}$$

In the next step, write slope-deflection equations for each span. For the span AB , B is below A and hence the chord joining AB' rotates in the clockwise direction (see Fig.15.3c)

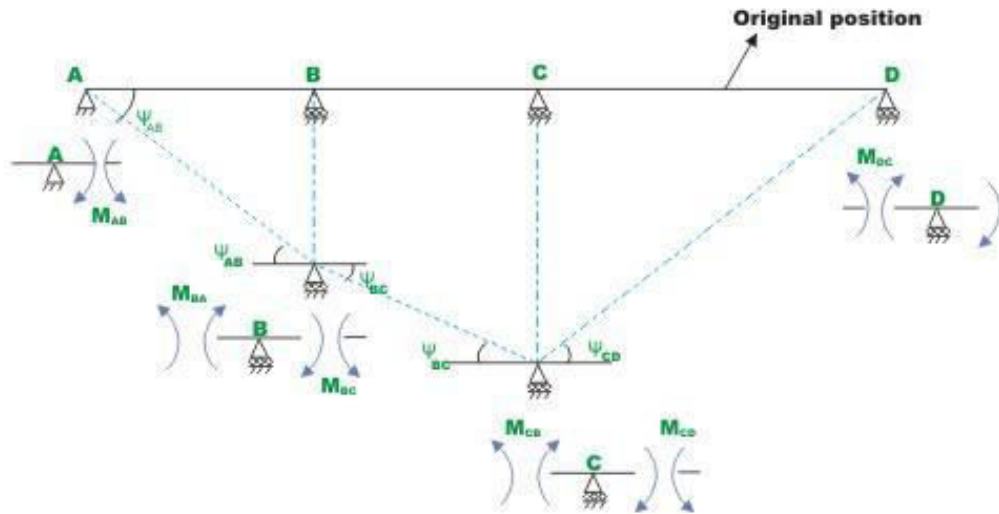


Fig 15.3c New support positions and free body diagrams of support

Now, writing the expressions for the span end moments, for each of the spans,

$$M_{AB} = 41.667 + 0.2EI (2\vartheta_A + \vartheta_B + 0.0005)$$

$$M_{BA} = -41.667 + 0.2EI (2\vartheta_B + \vartheta_A + 0.0005)$$

$$M_{BC} = 41.667 + 0.2EI (2\vartheta_B + \vartheta_C + 0.0005)$$

$$M_{CB} = -41.667 + 0.2EI (2\vartheta_C + \vartheta_B + 0.0005)$$

$$M_{CD} = 41.667 + 0.2EI (2\vartheta_C + \vartheta_D - 0.001)$$

$$M_{DC} = -41.667 + 0.2EI (2\vartheta_D + \vartheta_C - 0.001) \quad (3)$$

For the present problem, four joint equilibrium equations can be written, one each for each of the supports. They are

$$1. \quad M_A = 0 \Rightarrow M_{AB} = 0$$

$$2. \quad M_B = 0 \Rightarrow M_{BA} + M_{BC} = 0$$

$$3. \quad M_C = 0 \Rightarrow M_{CB} + M_{CD} = 0$$

$$\sum M_D = 0 \Rightarrow M_{DC} = 0 \quad (4)$$

Substituting the values of beam end moments from equations (3) in equation (4), four equations are obtained in four unknown rotations $\vartheta_A, \vartheta_B, \vartheta_C$ and ϑ_D . They are,

$$(EI = 200 \times 10^3 \times 1.35 \times 10^{-6} = 270,000 \text{ kN.m}^2)$$

$$2\vartheta_A + \vartheta_B = -1.2716 \times 10^{-3}$$

$$\vartheta_A + 4\vartheta_B + \vartheta_C = -0.001$$

$$\vartheta_B + 4\vartheta_C + \vartheta_D = 0.0005$$

$$\vartheta_C + 2\vartheta_D = 1.7716 \times 10^{-3}$$

Solving the above sets of simultaneous equations, values of ϑ_A , ϑ_B , ϑ_C and ϑ_D are evaluated.

Substituting the values in slope-deflection equations the beam end moments are evaluated.

$$M_{AB} = 41.667 + 0.2 \times 270,000\{2(-5.9629 \times 10^{-4}) + (-7.9013 \times 10^{-5}) + 0.0005\} = 0$$

$$M_{BA} = -41.667 + 0.2 \times 270,000\{2(-7.9013 \times 10^{-5}) - 5.9629 \times 10^{-4} + 0.0005\} = -55.40 \text{ kN.m}$$

$$M_{BC} = 41.667 + 0.2 \times 270,000\{2(-7.9013 \times 10^{-5}) + (-8.7653 \times 10^{-5}) + 0.0005\} = 55.40 \text{ kN.m}$$

$$M_{CB} = -41.667 + 0.2 \times 270,000\{2(-8.765 \times 10^{-5}) - 7.9013 \times 10^{-5} + 0.0005\} = -28.40 \text{ kN.m}$$

$$M_{CD} = 41.667 + 0.2 \times 270,000\{2 \times (-8.765 \times 10^{-5}) + 9.2963 \times 10^{-4} - 0.001\} = 28.40 \text{ kN.m}$$

$$M_{DC} = -41.667 + 0.2 \times 270,000\{2 \times 9.2963 \times 10^{-4} - 8.7653 \times 10^{-5} - 0.001\} = 0 \text{ kN.m (7)}$$

Reactions are obtained from equilibrium equations. Now consider the free body diagram of the beam with end moments and external loads as shown in Fig.15.3d.

$$R_A = 19.46 \text{ kN}(\uparrow)$$

$$R_{BL} = 30.54 \text{ kN}(\uparrow)$$

$$R_{BR} = 27.7 \text{ kN}(\uparrow)$$

$$R_{CL} = 22.3 \text{ kN}(\uparrow)$$

$$R_{CR} = 27.84 \text{ kN}(\uparrow)$$

$$R_D = 22.16 \text{ kN}(\uparrow)$$

The shear force and bending moment diagrams are shown in Fig.15.5e.

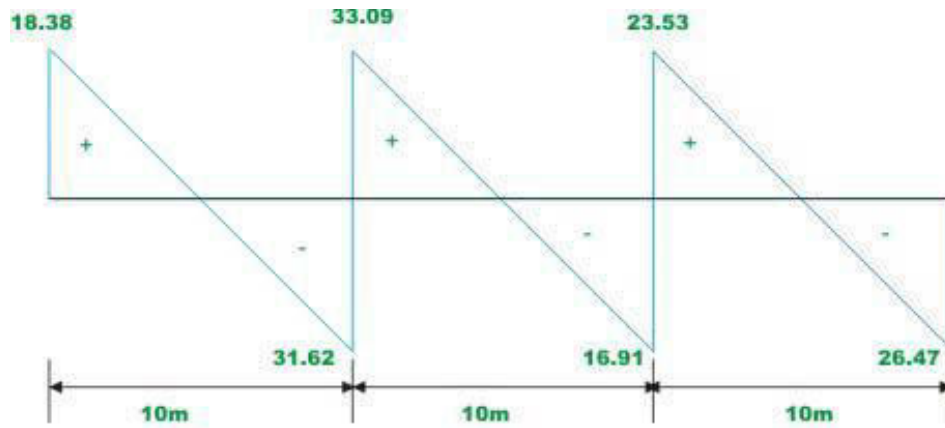


Fig 15.3d Shear force diagram

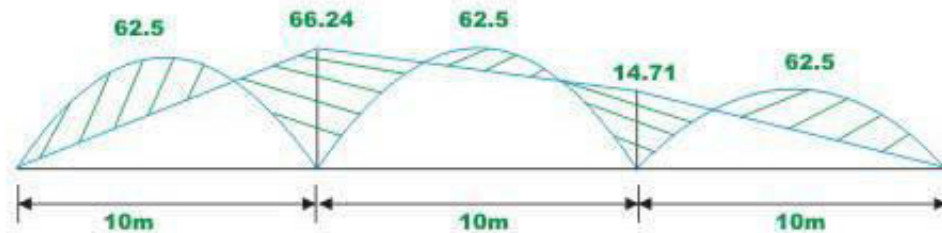


Fig. 15.3e Bending moment diagram

Summary

In this lesson, slope-deflection equations are derived for the case of beam with yielding supports. Moments developed at the ends are related to rotations and support settlements. The equilibrium equations are written at each support. The continuous beam is solved using slope-deflection equations. The deflected shape of the beam is sketched. The bending moment and shear force diagrams are drawn for the examples solved in this lesson.

Unit IV

Arches and Suspension Cables: Three hinged arches of different shapes, Eddy's Theorem, Suspension cable, stiffening girders, Two Hinged and Fixed Arches - Rib shortening and temperature effects

Three Hinged Arch

Introduction

In case of beams supporting uniformly distributed load, the maximum bending moment increases with the square of the span and hence they become uneconomical for long span structures. In such situations arches could be advantageously employed, as they would develop horizontal reactions, which in turn reduce the design bending moment.

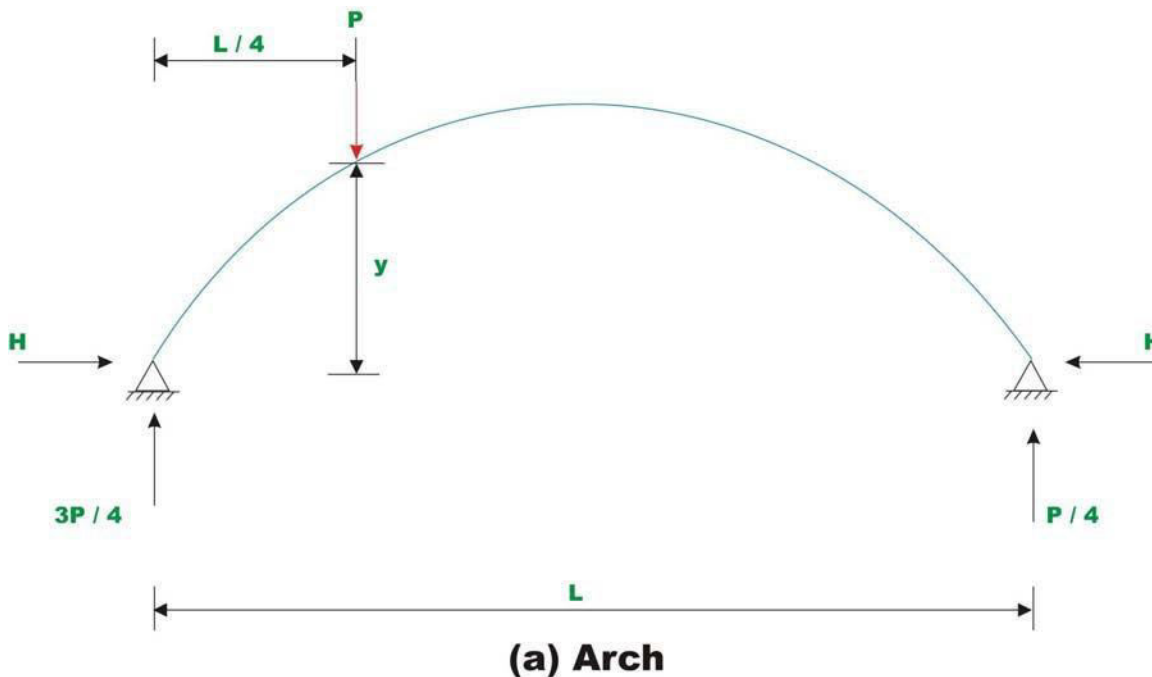


Fig. 32.1 Beam and Arch comparison.

For example, in the case of a simply supported beam shown in Fig. 32.1, the bending moment below the load is $3PL/16$. Now consider a two hinged symmetrical arch of the same span and subjected to similar loading as that of simply supported beam. The vertical reaction could be calculated by equations of statics. The horizontal reaction is determined by the method of least work. Now the bending moment below the load is $(3PL/16) Hy$. It is clear that the bending moment below the load is reduced in the case of an arch as compared to a simply supported beam. It is observed in the last lesson that, the cable takes the shape of the loading and this shape is termed as funicular shape. If an arch were constructed in an inverted funicular shape then it would be subjected to only compression for those loadings for which its shape is inverted funicular.

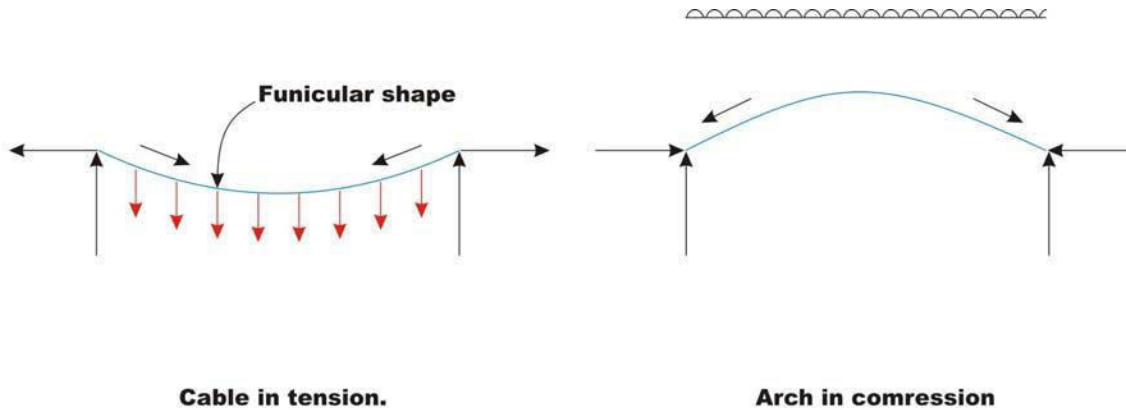


Fig. 32.2 Cable and Arch structure.

Since in practice, the actual shape of the arch differs from the inverted funicular shape or the loading differs from the one for which the arch is an inverted funicular, arches are also subjected to bending moment in addition to compression. As arches are subjected to compression, it must be designed to resist buckling.

Until the beginning of the 20th century, arches and vaults were commonly used to span between walls, piers or other supports. Now, arches are mainly used in bridge construction and doorways. In earlier days arches were constructed using stones and bricks. In modern times they are being constructed of reinforced concrete and steel.

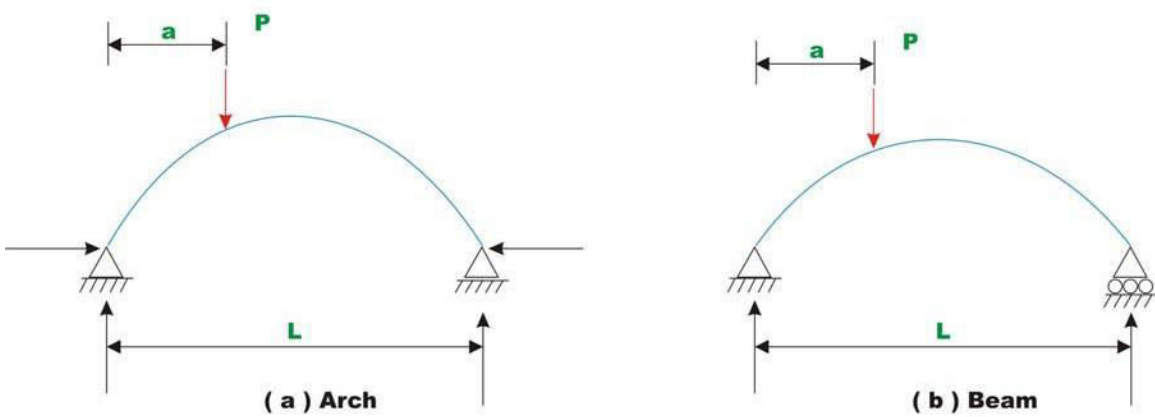


Fig. 32.3

A structure is classified as an arch not based on its shape but the way it supports the lateral load. Arches support load primarily in compression. For example in Fig 32.3b, no horizontal reaction is developed. Consequently bending moment is not reduced. It is important to appreciate the point that the definition of an arch is a structural one, not geometrical.

Type of arches

There are mainly three types of arches that are commonly used in practice: three hinged arch, two-hinged arch and fixed-fixed arch. Three-hinged arch is statically determinate structure and its reactions / internal forces are evaluated by static equations of equilibrium. Two-hinged arch and fixed-fixed arch are statically indeterminate structures. The indeterminate reactions are determined by the method of least work or by the flexibility matrix method. In this lesson three- hinged arch is discussed.

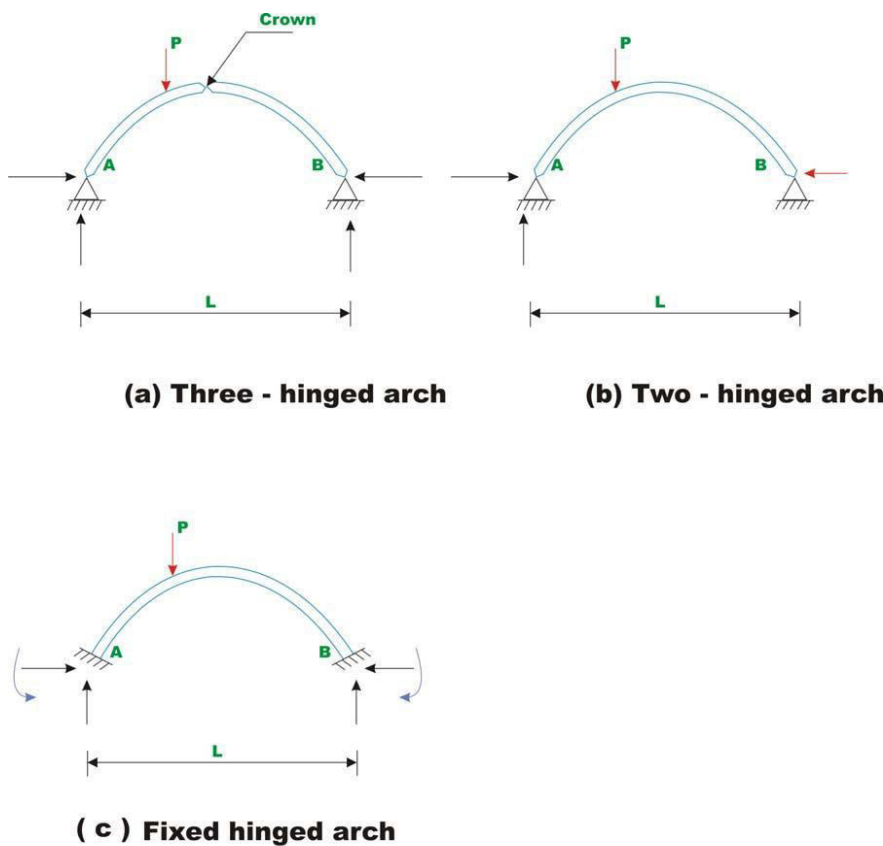


Fig. 32.4 Types of arches.

Analysis of three-hinged arch

In the case of three-hinged arch, we have three hinges: two at the support and one at the crown thus making it statically determinate structure. Consider a three hinged arch subjected

to a concentrated force P as shown in Fig 32.5.

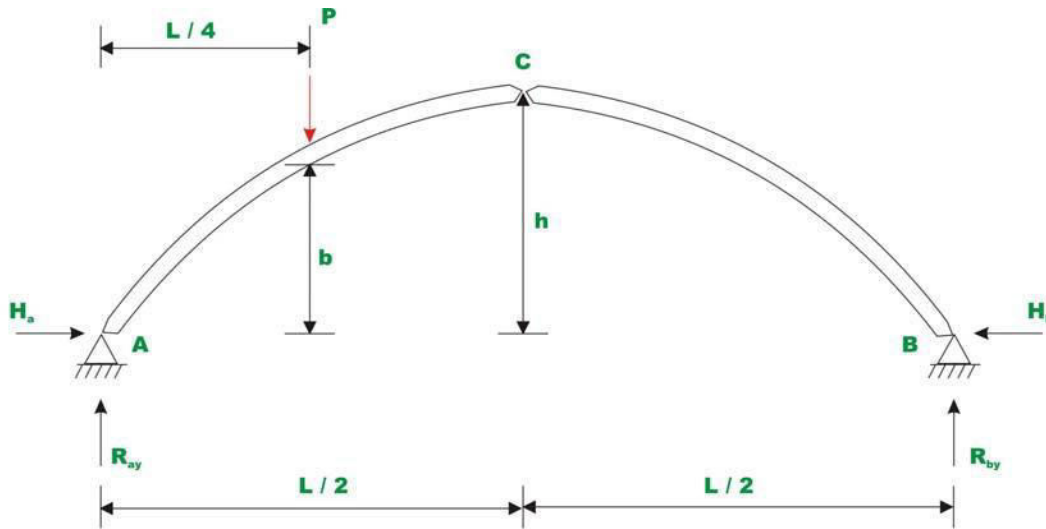


Fig. 32.5 Three hinged arch.

There are four reaction components in the three-hinged arch. One more equation is required in addition to three equations of static equilibrium for evaluating the four reaction components. Taking moment about the hinge of all the forces acting on either side of the hinge can set up the required equation.

$$H_a = H_b = PL/8h$$

$$V_a + V_b = \text{Total downwards loads}$$

Example 32.1

A three-hinged parabolic arch of uniform cross section has a span of 60 m and a rise of 10 m. It is subjected to uniformly distributed load of intensity 10 kN/m as shown in Fig. 32.6 Show that the bending moment is zero at any cross section of the arch.

Reactions:

Taking moment of all the forces about hinge A, yields

$$V_a = V_b = 10 \times 60 / 2 = 300 \text{ kN.}$$

Taking moment about left hinge c, we get

$$V_a \times 30 - 10 \times 30 \times 15 - H_a \times 10 = 0$$

$$H_a = H_b = 450 \text{ kN.}$$

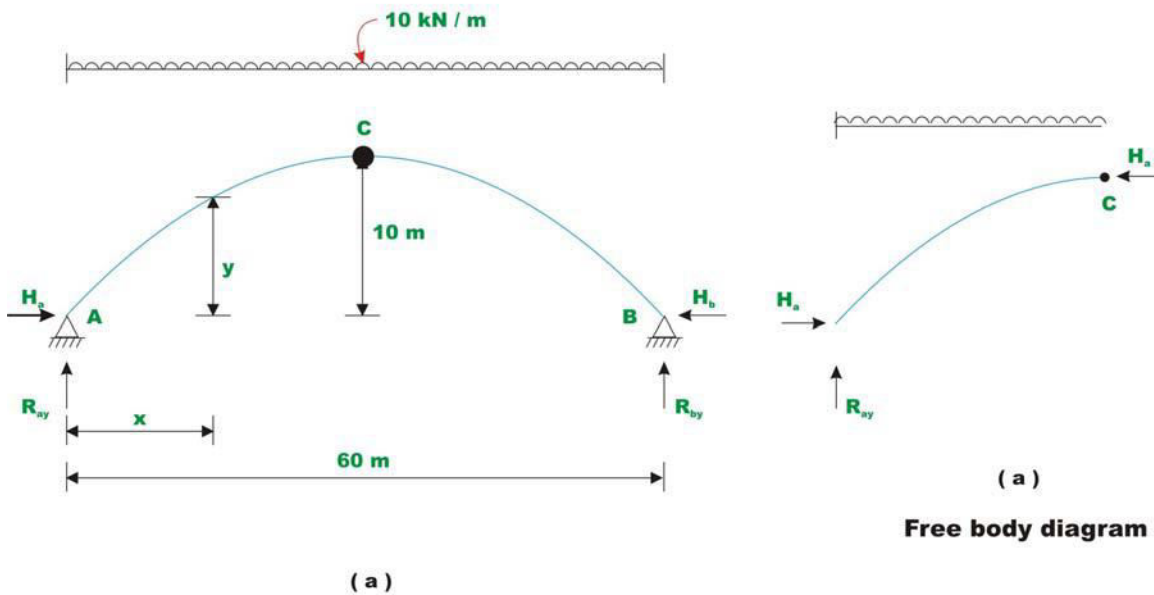


Fig. 32.6 Three hinged arch of Example 32.1

BM at any section XX is given as

$$BM_{xx} = V_a \cdot x - H_a \cdot y - 10 \cdot x \cdot (x/2)$$

The equation of the three-hinged parabolic arch is $y = 4hx(L-x)/L^2$ substituting

$$BM_{xx} = 300 \cdot x - 450(4 \cdot 10 \cdot x \cdot (60-x)) - 5x^2 = 300x - (450(2400x - 40x^2))/(60 \cdot 60) - 5x^2$$

$$= 300x - 300x + 5x^2 - 5x^2 = 0$$

In other words a three hinged parabolic arch subjected to uniformly distributed load is not subjected to bending moment at any cross section. It supports the load in pure compression.

Example 32.2

A three-hinged semicircular arch of uniform cross section is loaded as shown in Fig 32.7. Calculate the location and magnitude of maximum bending moment in the arch.

Solution:

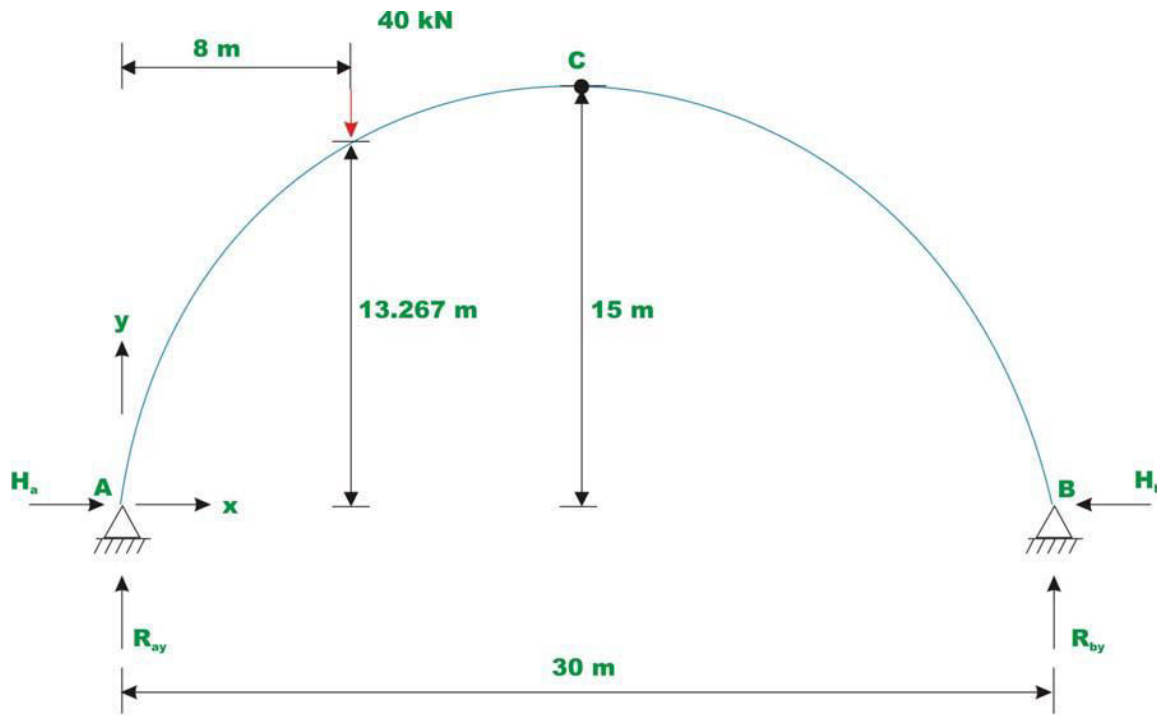


Fig. 32.7 A semi circular arch of Example 32.2

Reactions:

Taking moment of all the forces about hinge *B* leads to,

$$V_a = 29.33 \text{ kN} \quad V_b = 10.67 \text{ kN}.$$

Bending moment

Now making use of the condition that the moment at hinge of all the forces left of hinge is zero gives, *CC*

$$H_a = H_b = 10.66 \text{ kN}.$$

The maximum positive bending moment occurs below *D* and it can be calculated by taking moment of all forces left of about *D*,

$$Y \text{ at } D = \frac{4 \times 15 \times 8(30-8)}{30^2} = 13.267 \text{ m}.$$

$$\text{BM at } D = V_a \times 8 - H_a \times y = 93.213 \text{ kN M}.$$

Example 32.3

A three-hinged parabolic arch is loaded as shown in Fig 32.8a. Calculate the location and magnitude of maximum bending moment in the arch. Draw bending moment diagram.

Solution:

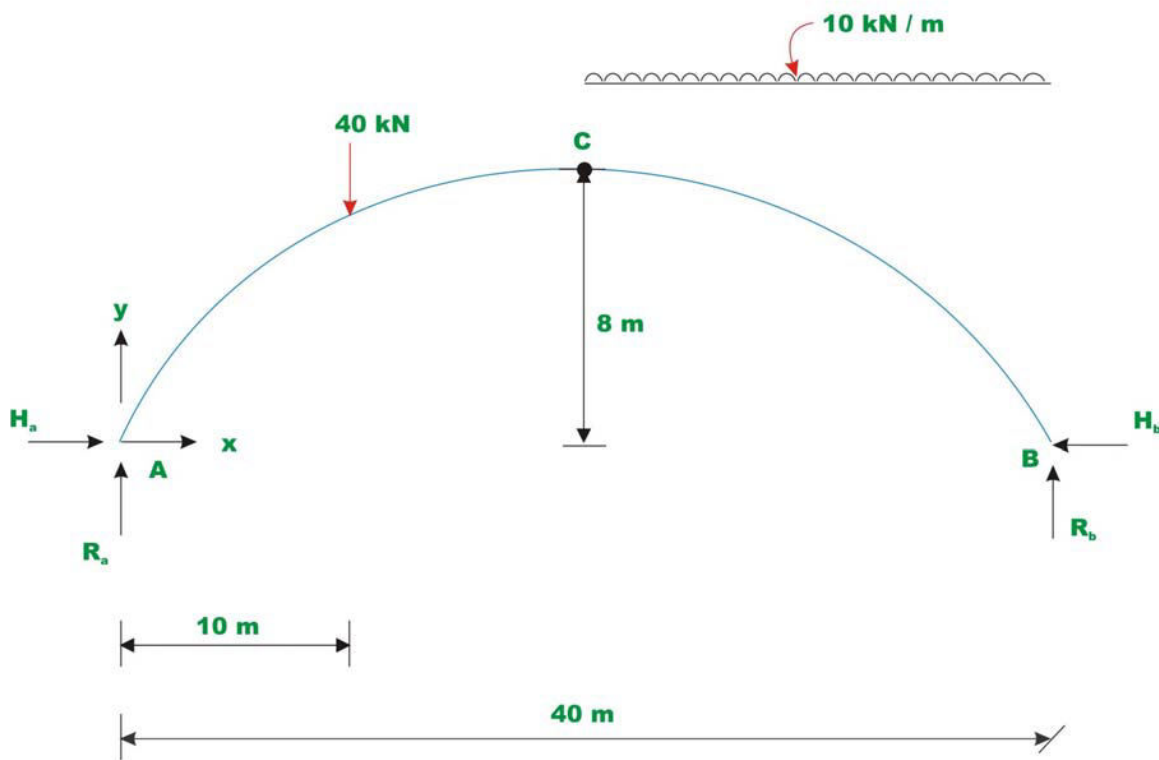


Fig. 32.8a Example 32.3

Reactions:

Taking A as the origin, the equation of the three-hinged parabolic arch is given by,

$$V_a = 80.0 \text{ kN} \quad V_b = 160.0 \text{ kN}.$$

$$H_a = H_b = 150.0 \text{ kN}.$$

Location of maximum bending moment

Consider a section x from end B. Moment at section x in part CB of the arch is given by (please note that B has been taken as the origin for this calculation),

$$BM = 160x - 150y - 10x^2/2$$

According to calculus, the necessary condition for extreme (maximum or minimum) is that

$$d(BM)/dx = 0 \text{ solving we get } x = 10.0 \text{ m}.$$

$$BM \text{ max} = 200 \text{ kN m}.$$

Shear force at D just left of 40 kN load

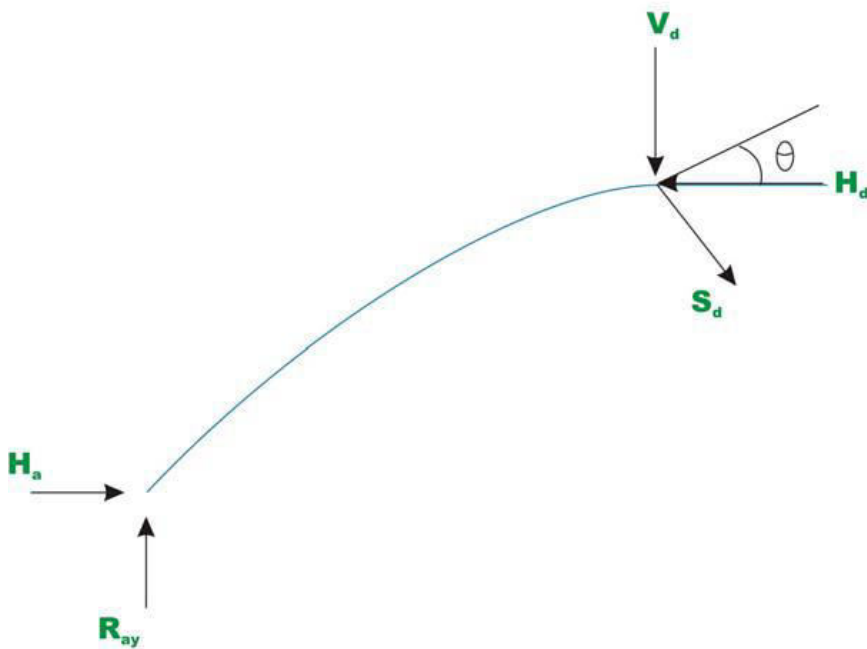


Fig. 32.8b

The slope of the arch at D is evaluated by, $\tan \theta = dy/dx = (8/10) - (16/400)x$

Put $x = 10.0$ m and solving $\theta = 21.8^\circ$

Shear force at left of D is $= H_a \sin \theta - V_a \cos \theta = -18.57$ kN.

Example 32.4

A three-hinged parabolic arch of constant cross section is subjected to a uniformly distributed load over a part of its span and a concentrated load of 50 kN, as shown in Fig. 32.9. The dimensions of the arch are shown in the figure. Evaluate the horizontal thrust and the maximum bending moment in the arch.

Solution:

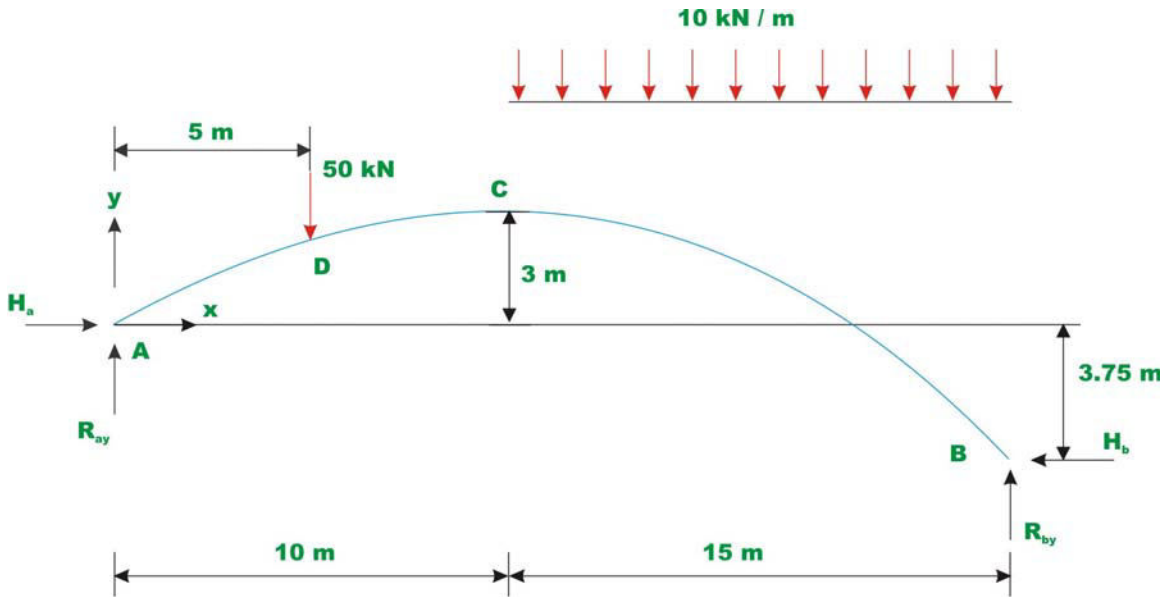


Fig. 32.9

Reactions:

Taking A as the origin, the equation of the parabolic arch may be written as,

$$Y = 4hx(L-x)/L^2 = 4 \times 3 \times x(20-x)/20^2 = -0.03x^2 + 0.6x$$

Taking moment of all the loads about B leads to,

$$V_a \times 25 + H_a \times 3.75 - 50 \times 20 - 10 \times 15 \times 7.5 = 0$$

$$V_a = (2125 - 3.75H_a)/25$$

Taking moment of all the forces right of hinge C about the hinge C and setting leads to,

$$V_b = (1125 + 6.75 H_b)/15$$

$$H_a = H_b$$

$$V_a + V_b = 50 + 10 \times 15 = 200 \text{ kN.}$$

Substituting and solving $H_a = H_b = 133.33 \text{ kN.}$

$$V_a = 65.0 \text{ kN, } V_b = 135.0 \text{ kN.}$$

Bending moment

From inspection, the maximum negative bending moment occurs in the region *AD* and the maximum positive bending moment occurs in the region *CB*.

Span AD

Bending moment at any cross section in the span AD is

$$BM = V_a * x - H_a * y = 65 * x - 133.33 (-0.03x^2 + 0.6x), x \text{ lies between } 0 \text{ to } 5$$

For, the maximum negative bending moment in this region, $dB M / dx = 0$

Solving $x = 1.8748 \text{ m}$.

$$BM = -14.06 \text{ kN M.}$$

For the maximum positive bending moment in this region occurs at D,

$$BM = V_b * 5 - H_b * y = 135 * 5 - 133.33 * (-0.03 * 5^2 + 0.6 * 5) = 25.0 \text{ kN M.}$$

Span CB

Bending moment at any cross section, in this span is calculated by,

$$BM = V_a * x - H_a * (-0.03x^2 + 0.6x) - 50(x - 5) - 10(x - 10)(x - 5)/2$$

For locating the position of maximum bending moment, $dB M / dx = 0$

Solving $x = 17.5 \text{ m}$.

$$BM = 56.25 \text{ kN M.}$$

Hence, the maximum positive bending moment occurs in span CB.

Summary

In this lesson, the arch definition is given. The advantages of arch construction are given in the introduction. Arches are classified as three-hinged, two-hinged and hinge less arches. The analysis of three-hinged arch is considered here. Numerical examples are solved in detail to show the general procedure of three-hinged arch analysis.

Two-Hinged Arch

Introduction

Mainly three types of arches are used in practice: three-hinged, two-hinged and hinge less arches. In the early part of the nineteenth century, three-hinged arches were commonly used for the long span structures as the analysis of such arches could be done with confidence. However, with the development in structural analysis, for long span structures starting from late

nineteenth century engineers adopted two-hinged and hinge less arches. Two-hinged arch is the statically indeterminate structure to degree one. Usually, the horizontal reaction is treated as the redundant and is evaluated by the method of least work. In this lesson, the analysis of two-hinged arches is discussed and few problems are solved to illustrate the procedure for calculating the internal forces.

Analysis of two-hinged arch

A typical two-hinged arch is shown in Fig. 33.1a. In the case of two-hinged arch, we have four unknown reactions, but there are only three equations of equilibrium available. Hence, the degree of statical indeterminacy is one for two-hinged arch.

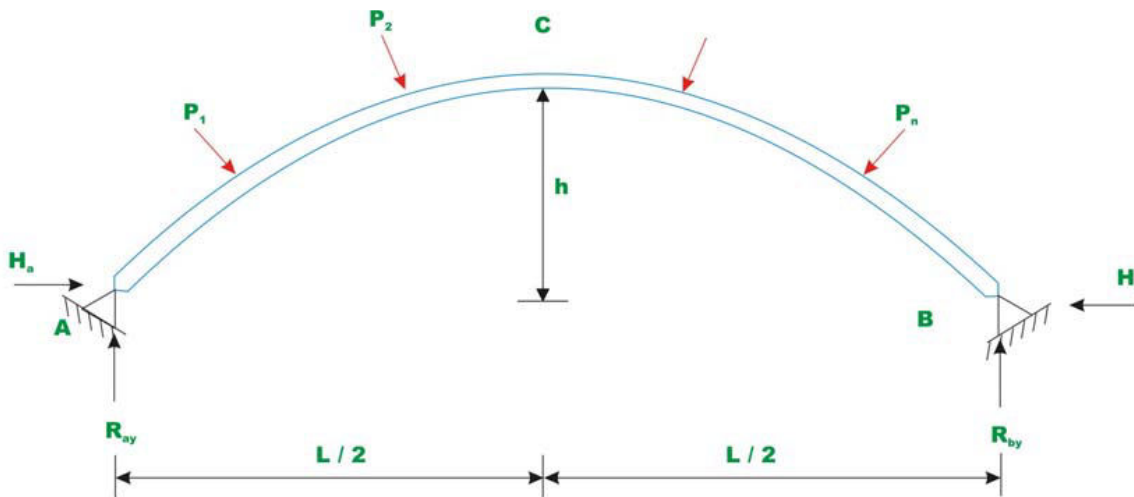


Fig. 33.1a Two - hinged arch.

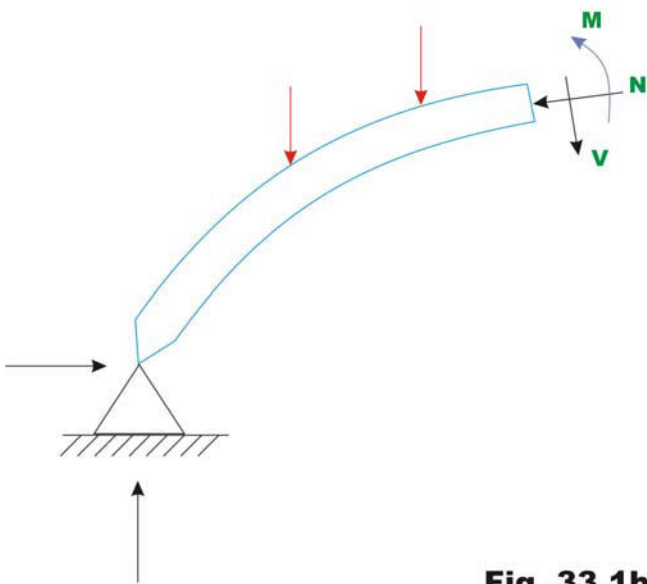


Fig. 33.1b

The fourth equation is written considering deformation of the arch. The unknown redundant reaction H_b is calculated by noting that the horizontal displacement of hinge bHB is zero. In general the horizontal reaction in the two hinged arch is evaluated by straightforward application of the theorem of least work (see module 1, lesson 4), which states that the partial derivative of the strain energy of a statically indeterminate structure with respect to statically indeterminate action should vanish. Hence to obtain, horizontal reaction, one must develop an expression for strain energy. Typically, any section of the arch (vide Fig 33.1b) is subjected to shear force V , bending moment M and the axial compression. The strain energy U_b due to bending is calculated from the following expression.

$$U_b = \int_0^s (M^2/2EI) ds$$

The above expression is similar to the one used in the case of straight beams. However, in this case, the integration needs to be evaluated along the curved arch length. In the above equation, s is the length of the centerline of the arch, I is the moment of inertia of the arch cross section, E is the Young's modulus of the arch material. The strain energy due to shear is small as compared to the strain energy due to bending and is usually neglected in the analysis. In the case of flat arches, the strain energy due to axial compression can be appreciable and is given by,

$$U_a = \int_0^s (N^2/2AE) ds$$

The total strain energy of the arch is given by,

$$U = \int_0^s (M^2/2EI) ds + \int_0^s (N^2/2AE) ds$$

Now, according to the principle of least work $\partial U/\partial H = 0$, where H is chosen as the redundant reaction.

$$\partial U/\partial H = \int_0^s (M/EI) (\partial M/\partial H) ds + \int_0^s (N/AE) (\partial N/\partial H) ds$$

Solving above equation, the horizontal reaction H is evaluated.

Symmetrical two hinged arch

Consider a symmetrical two-hinged arch as shown in Fig 33.2a. Let at crown be the origin of co-ordinate axes. Now, replace hinge at B with a roller support. Then we get a simply supported curved beam as shown in Fig 33.2b. Since the curved beam is free to move horizontally, it will do so as shown by dotted lines in Fig 33.2b. Let M_0 and N_0 be the bending moment and axial force at any cross section of the simply supported curved beam. Since, in the original arch structure, there is no horizontal displacement, now apply a horizontal force H as shown in Fig. 33.2c. The horizontal force H should be of such magnitude, that the displacement at B must vanish.

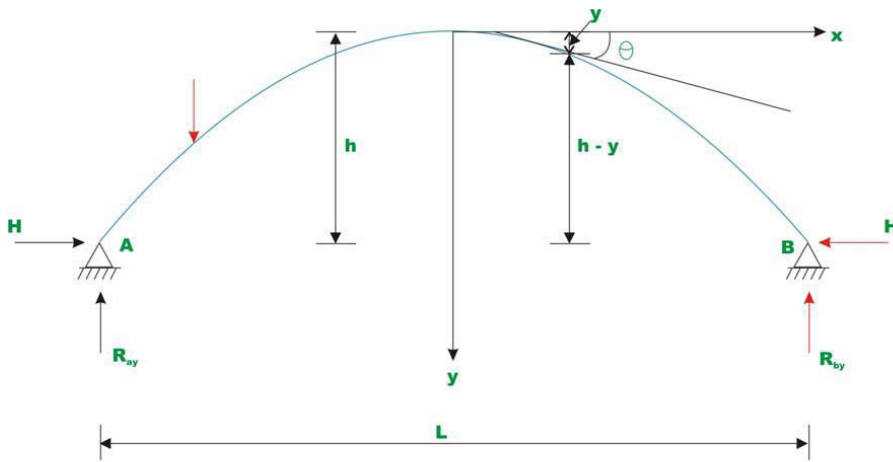


Fig. 33.2a

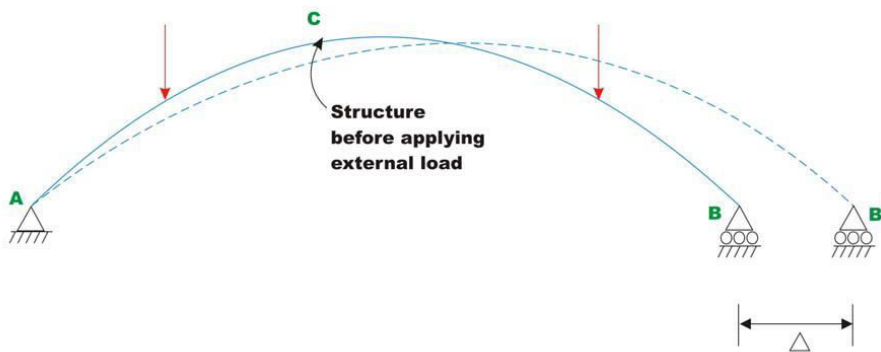


Fig. 33.2b.

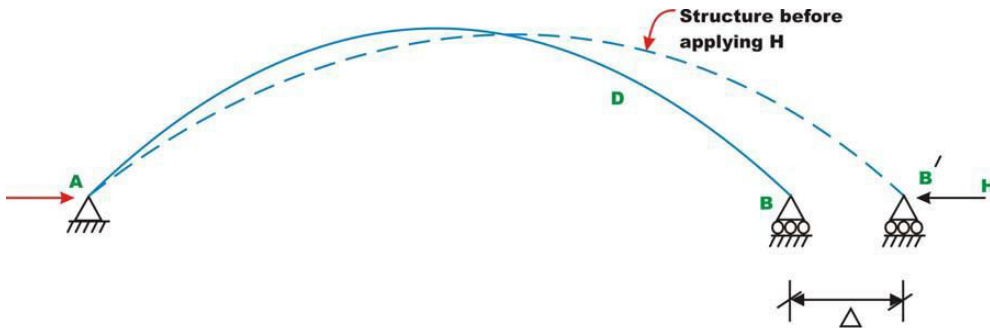


Fig. 33.2c.

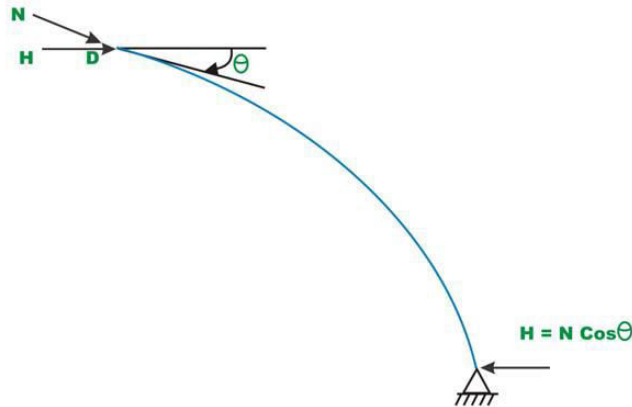


Fig. 33.2d.

From Fig. 33.2b and Fig 33.2c, the bending moment at any cross section of the arch (say), may be written as

$$M = M_o - H(h - y)$$

The axial compressive force at any cross section (say) may be written as

$$N = N_o + H \cos \theta$$

Where θ is the angle made by the tangent at with horizontal (vide Fig 33.2d). D

Substituting the value of M and in the equation (33.4),

$$\partial U / \partial H = 0 = -\int_0^s (M_o - H(h - y)(h - y) / EI) ds + \int_0^s (N_o + H \cos \theta \cos \theta / AE) ds$$

Solving for H , yields,

$$H = [\int_0^s (M_o / EI) y' ds] / [\int_0^s (y'^2 / AE) ds]$$

For an arch with uniform cross section EI is constant and hence,

$$H = [\int_0^s (M_o) y' ds] / [\int_0^s (y'^2) ds]$$

In the above equation, M_o is the bending moment at any cross section of the arch when one of the hinges is replaced by a roller support. Y' is the height of the arch as shown in the figure. If the

moment of inertia of the arch rib is not constant, then equation (33.10) must be used to calculate the horizontal reaction H .

Temperature effect

Consider an unloaded two-hinged arch of span L . When the arch undergoes a uniform temperature change of T °C, then its span would increase by $TL\alpha$ if it were allowed to expand freely (vide Fig 33.3a). α is the co-efficient of thermal expansion of the arch material. Since the arch is restrained from the horizontal movement, a horizontal force is induced at the support as the temperature is increased.

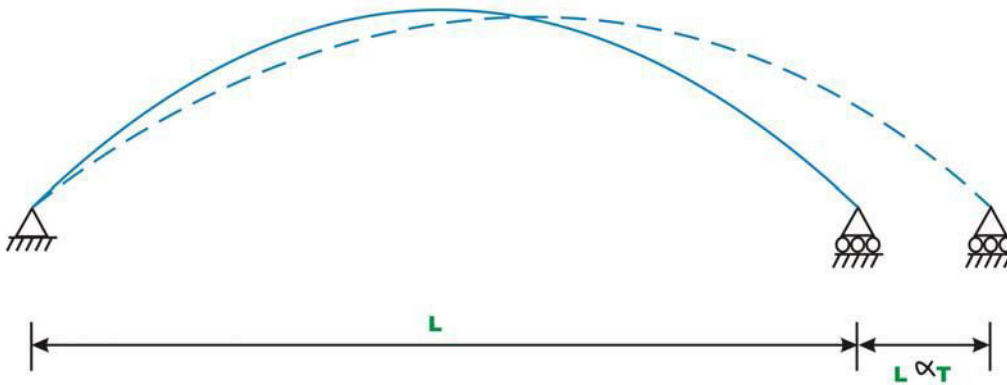


Fig. 33.3a

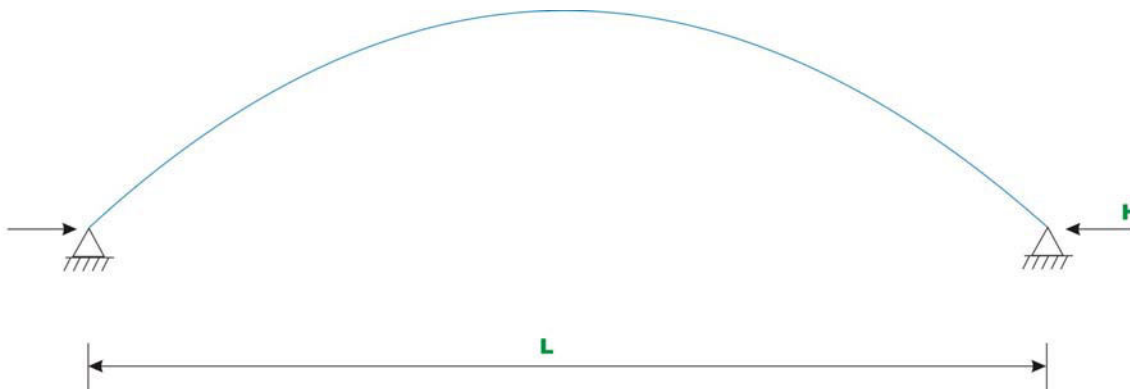


Fig. 33.3b.

Now applying the Castigliano's first theorem,

$$\partial U / \partial H = TL\alpha$$

Solving for H ,

$$H = [TL\alpha] / [\int_0^s (y'^2 / EI) ds]$$

Example 33.1

A semicircular two hinged arch of constant cross section is subjected to a concentrated load as shown in Fig 33.4a. Calculate reactions of the arch and draw bending moment diagram.

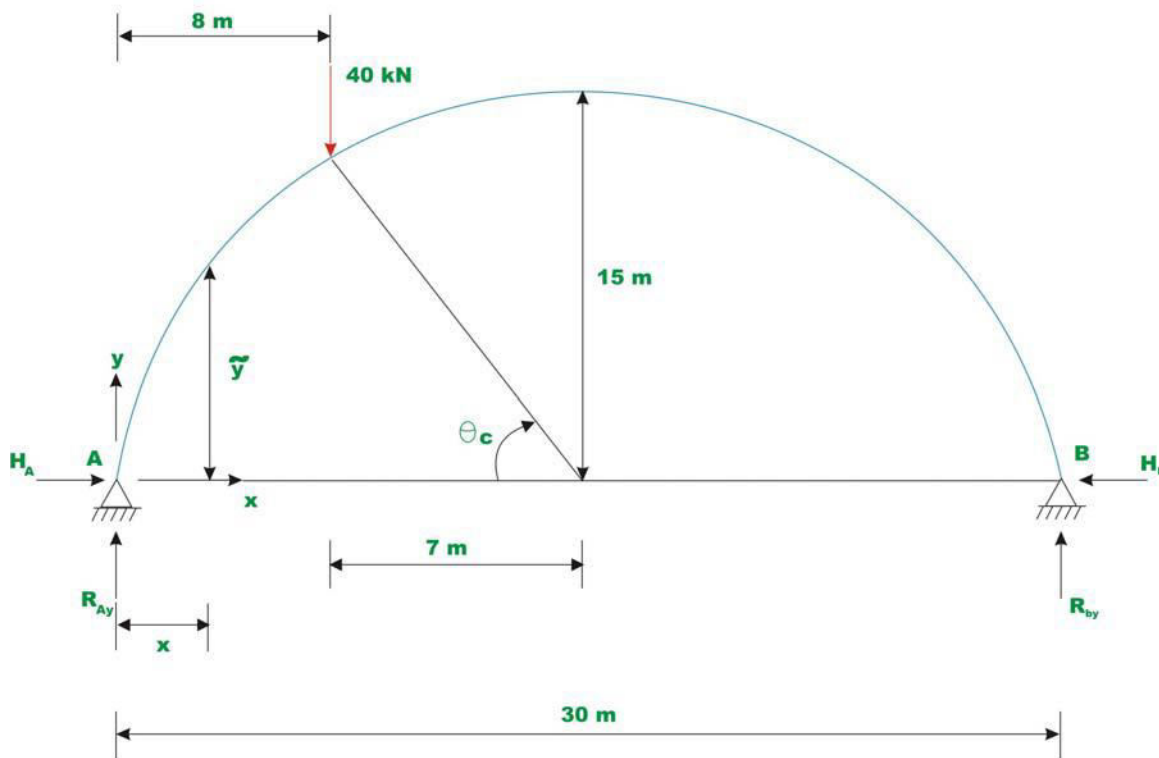


Fig. 33.4a.

Solution:

Taking moment of all forces about hinge B leads to,

$$V_a = 29.33 \text{ kN.}$$

$$V_b = 10.67 \text{ kN.}$$

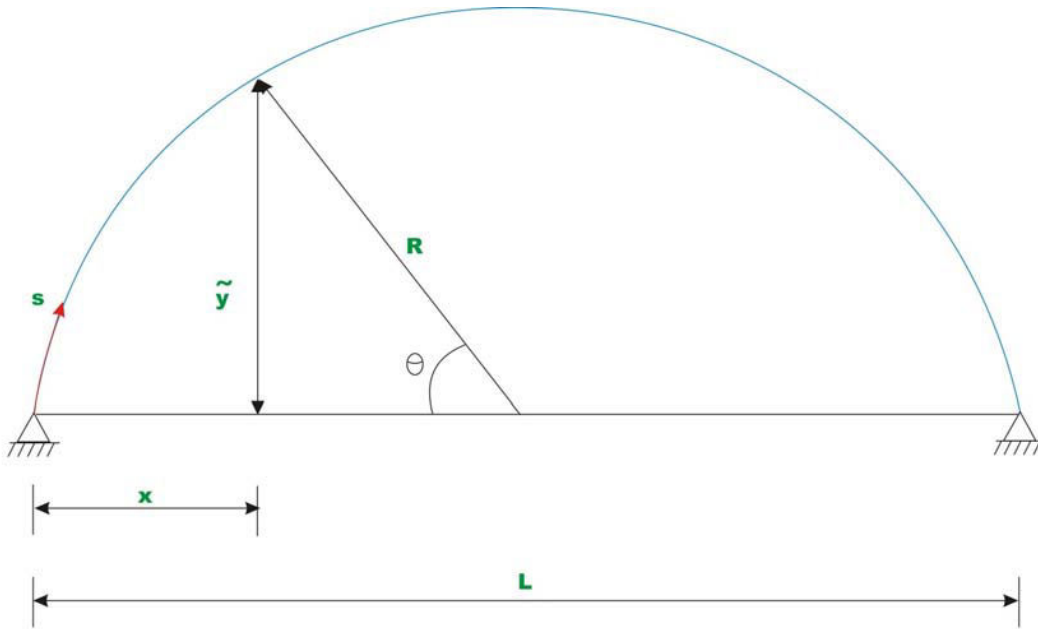


Fig. 33.4b.

From Fig. 33.4b,

$$Y' = R \sin \theta$$

$$X = R (1 - \cos \theta)$$

$$Ds = R d \theta$$

$$\tan \theta_c = 13.267/7$$

$$\theta_c = 62.10^\circ$$

Now, the horizontal reaction H may be calculated by the following expression,

$$H = \left[\int_0^s (M_o) y' ds \right] / \left[\int_0^s (y'^2) ds \right]$$

Now M_o is the bending moment at any cross section of the arch when one of the hinges is replaced by a roller support is given by,

$$M_o = V_a \cdot R (1 - \cos \theta) \text{ for } \theta_c \text{ lying between } 0 \text{ to } \theta$$

$$M_o = V_a \cdot R (1 - \cos \theta) - 40 (x - 8) \text{ for } \theta_c \text{ lying between } \theta \text{ to } \pi$$

Integrating and solving, the horizontal thrust at the support is,

$$H = 19.90 \text{ kN.}$$

Bending moment diagram

Bending moment M at any cross section of the arch is given by,

$M = M_o - H_y'$ the bending moment diagram is shown in Fig. 33.4c.

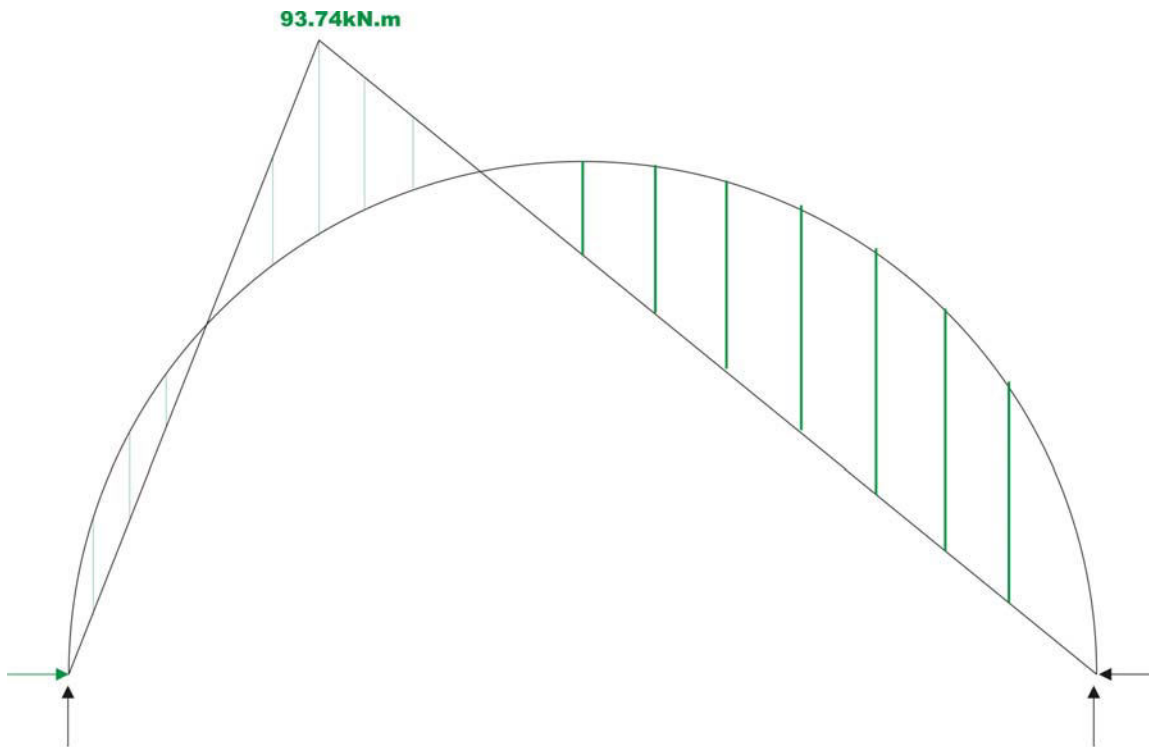


Fig. 33.4c Bending moment diagram

Summary

Two-hinged arch is the statically indeterminate structure to degree one. Usually, the horizontal reaction is treated as the redundant and is evaluated by the method of least work. Towards this end, the strain energy stored in the two-hinged arch during deformation is given. The reactions developed due to thermal loadings are discussed. Finally, a few numerical examples are solved to illustrate the procedure.

Cables

Introduction

Cables and arches are closely related to each other and hence they are grouped in this course in the same module. For long span structures (for e.g. in case bridges) engineers commonly use cable or arch construction due to their efficiency. In the first lesson of this module, cables subjected to uniform and concentrated loads are discussed. In the second lesson, arches in general and three hinged arches in particular along with illustrative examples are explained. In the last two lessons of this module, two hinged arch and hinge less arches are considered.

Structure may be classified into rigid and deformable structures depending on change in geometry of the structure while supporting the load. Rigid structures support externally applied loads without appreciable change in their shape (geometry). Beams trusses and frames are examples of rigid structures. Unlike rigid structures, deformable structures undergo changes in their shape according to externally applied loads. However, it should be noted that deformations are still small. Cables and fabric structures are deformable structures. Cables are mainly used to support suspension roofs, bridges and cable car system. They are also used in electrical transmission lines and for structures supporting radio antennas. In the following sections, cables subjected to concentrated load and cables subjected to uniform loads are considered.

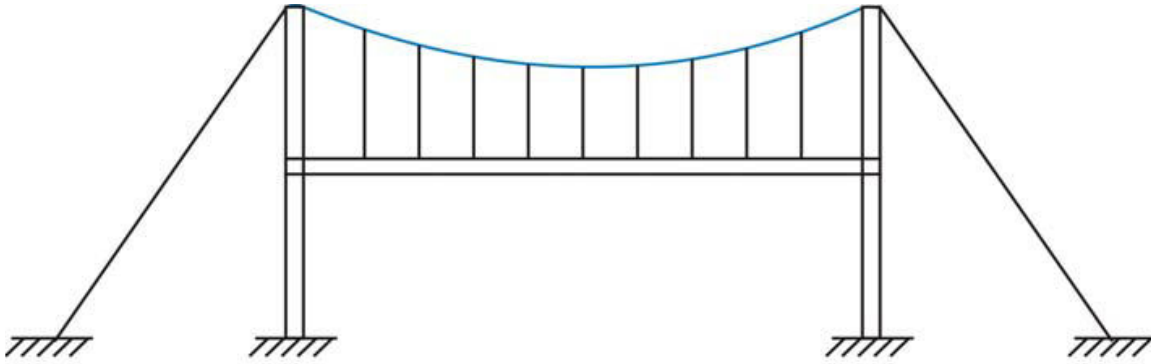


Fig. 31.1 Deformable structure.



**Fig 31.2a Unloaded cable
(when dead load is neglected)**

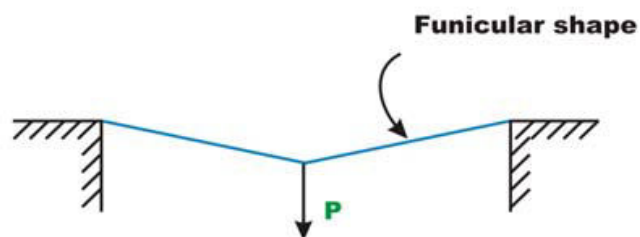


Figure 31.2b Cable in tension.

The shape assumed by a rope or a chain (with no stiffness) under the action of external loads when hung from two supports is known as a funicular shape. Cable is a funicular structure. It is

easy to visualize that a cable hung from two supports subjected to external load must be in tension (vide Fig. 31.2a and 31.2b). Now let us modify our definition of cable. A cable may be defined as the structure in pure tension having the funicular shape of the load.

Cable subjected to Concentrated Loads

As stated earlier, the cables are considered to be perfectly flexible (no flexural stiffness) and inextensible. As they are flexible they do not resist shear force and bending moment. It is subjected to axial tension only and it is always acting tangential to the cable at any point along the length. If the weight of the cable is negligible as compared with the externally applied loads then its self-weight is neglected in the analysis. In the present analysis self-weight is not considered.

Cable subjected to uniform load.

Cables are used to support the dead weight and live loads of the bridge decks having long spans. The bridge decks are suspended from the cable using the hangers. The stiffened deck prevents the supporting cable from changing its shape by distributing the live load moving over it, for a longer length of cable. In such cases cable is assumed to be uniformly loaded.

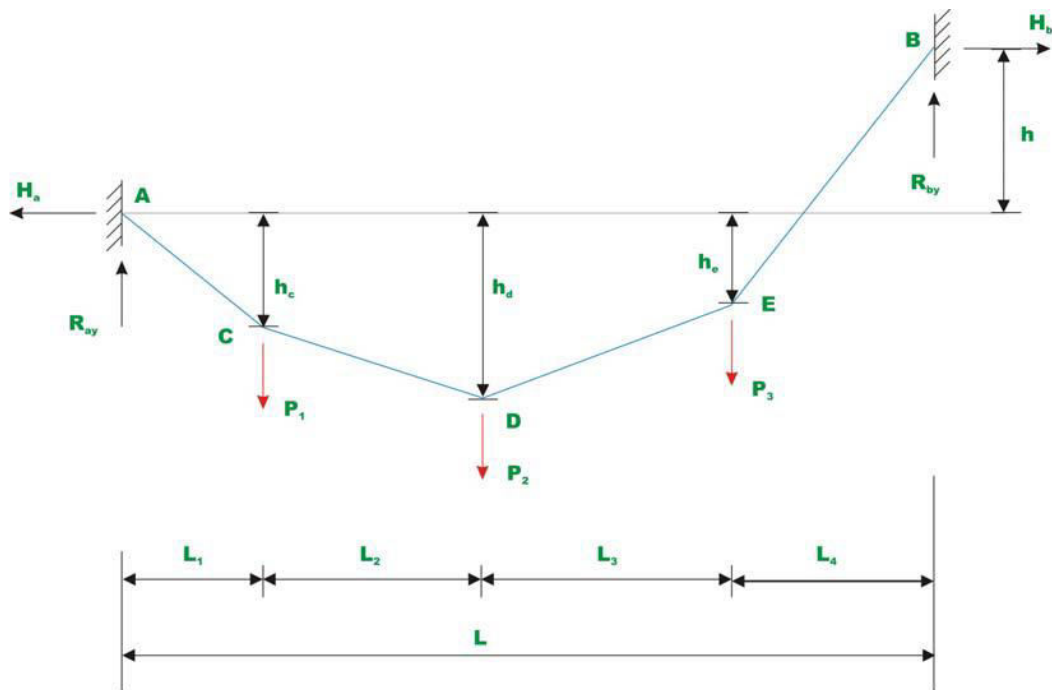


Fig. 31.3a Cable subjected to concentrated load.

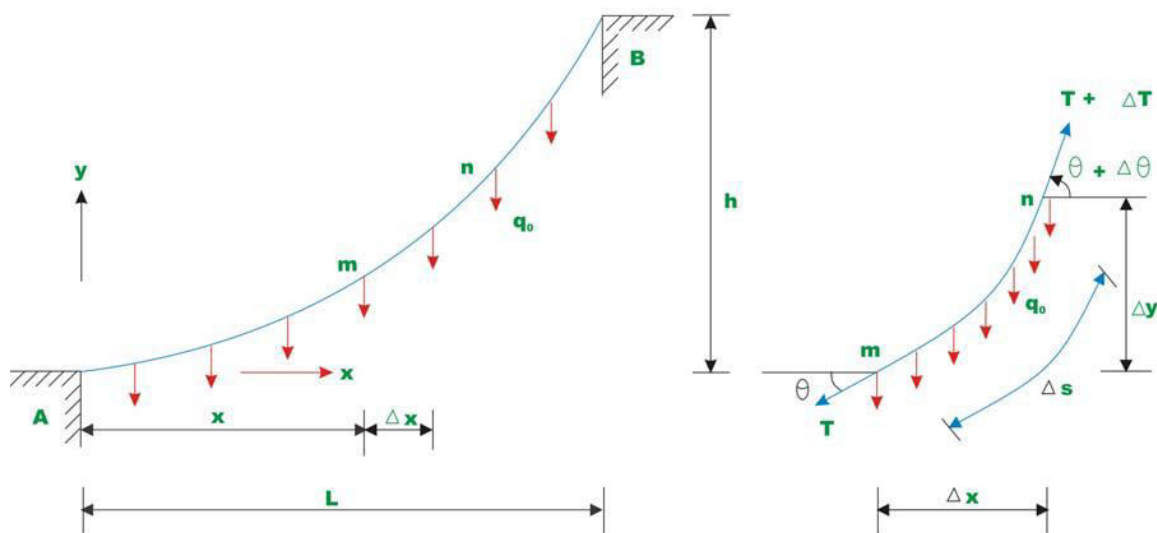


Fig. 31.3b Cable subjected to uniformly distributed load.

Fig. 31.3c Free-body diagram

Equation of cable $y = \frac{q_0 x^2}{2H}$

Equation represents a parabola. Now the tension in the cable may be evaluated as

$$T = \sqrt{q_0 x^2 + H^2}$$

$T = T_{\max}$ when $x = L$,

$$T_{\max} = \sqrt{q_0 L^2 + H^2}$$

Due to uniformly distributed load, the cable takes a parabolic shape. However due to its own dead weight it takes a shape of a catenary. However dead weight of the cable is neglected in the present analysis.

Summary

In this lesson, the cable is defined as the structure in pure tension having the funicular shape of the load. The procedures to analyse cables carrying concentrated load and uniformly distributed loads are developed. A few numerical examples are solved to show the application of these methods to actual problems.

UNIT-V

Rolling loads and Influence Lines: Maximum SF and BM curves for various types of rolling loads, focal length, EUDL, Influence Lines for Determinate Structures- Beams, Three Hinged Arches

INFLUENCE LINE DIAGRAMS

Introduction

In the previous lessons, we have studied about construction of influence line for the either single concentrated load or uniformly distributed loads. In the present lesson, we will study in depth about the beams, which are loaded with a series of two or more than two concentrated loads.

Maximum shear at sections in a beam supporting two concentrated loads

Let us assume that instead of one single point load, there are two point loads P_1 and P_2 spaced at y moving from left to right on the beam as shown in Figure 1.1. We are interested to find maximum shear force in the beam at given section C. In the present case, we assume that $P_2 < P_1$.

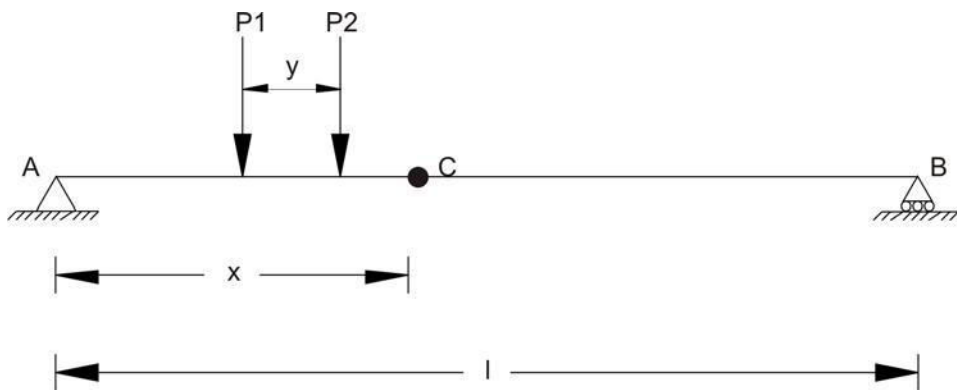


Figure 1.1: Beam loaded with two concentrated point loads

Now there are three possibilities due to load spacing. They are: $x < y$, $x = y$ and $x > y$.

Case 1: $x < y$

This case indicates that when load P_2 will be between A and C then load P_1 will not be on the beam. In that case, maximum negative shear at section C can be given by

$$V_C = -P_2 \times \frac{x}{l}$$

and maximum positive shear at section C will be

Case 2: $x = y$

In this case, load P_1 will be on support A and P_2 will be on section C. Maximum negative shear can be given by

$V_C = -P_2 \times \frac{x}{l}$ and maximum positive shear at section C will be

Case 3: $x > y$

With reference to Figure 1.2, maximum negative shear force can be obtained when load P_2 will be on section C. The maximum negative shear force is expressed as:

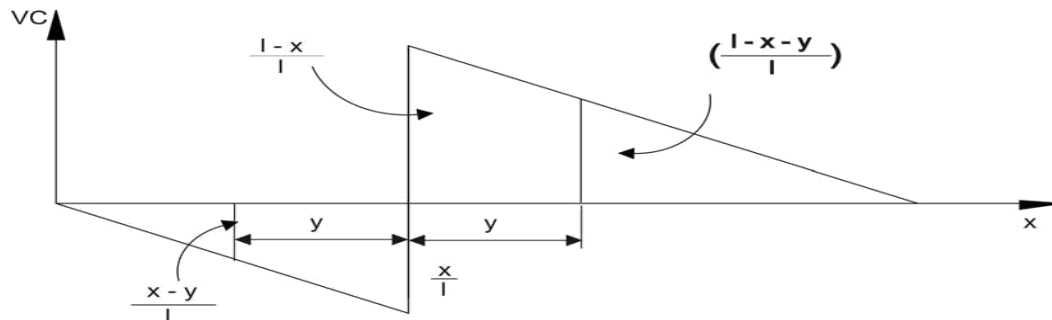


Figure 1.2: Influence line for shear at section C

$$V_C^1 = -P_2 x/l - P_1 (x-y)/l$$

And with reference to Figure 1.2, maximum positive shear force can be obtained when load P_1 will be on section C. The maximum positive shear force is expressed as:

$$V_C^2 = -P_1 x/l + P_2 (l-x-y)/l$$

From above discussed two values of shear force at section, select the maximum negative shear value.

Maximum moment at sections in a beam supporting two concentrated loads

Let us assume that instead of one single point load, there are two point loads P_1 and P_2 spaced at y moving left to right on the beam as shown in Figure 1.3. We are interested to find maximum moment in the beam at given section C.

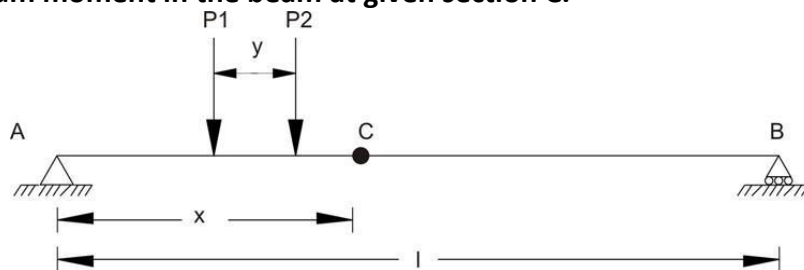


Figure 1.3: Beam loaded with two concentrated loads

With reference to Figure 1.4, moment can be obtained when load P_2 will be on section C. The moment for this case is expressed as:

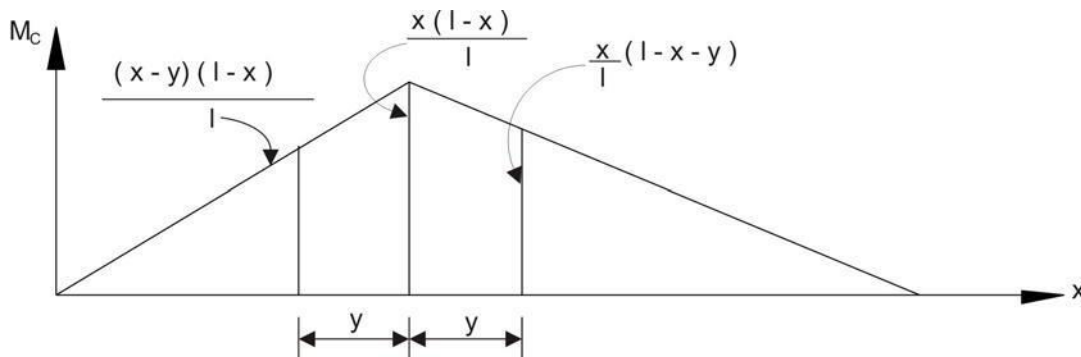


Figure 1.4: Influence line for moment at section C

$$M_C^1 = P_1 (x - y) \frac{l - x}{l} + P_2 x \frac{l - x}{l}$$

With reference to Figure 1.4, moment can be obtained when load P_1 will be on section C. The moment for this case is expressed as:

$$M_C^2 = P_1 x \frac{l - x - y}{l} + P_2 x \frac{l - x - y}{l}$$

From above two cases, maximum value of moment should be considered for maximum moment at section C when two point loads are moving from left end to right end of the beam.

Maximum end shear in a beam supporting a series of moving concentrated loads

In real life situation, usually there are more than two point loads, which will be moving on bridges. Hence, in this case, our aim is to learn, how to find end shear in beam supporting a series of moving concentrated loads. Let us assume that as shown in Figure 1.5, four concentrated loads are moving from right end to left end on beam AB. The spacing of the concentrated load is given in Figure 1.5.

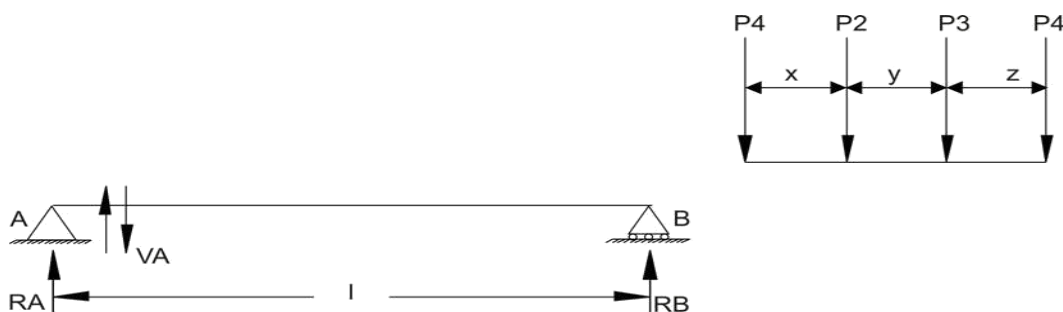


Figure 1.5: Beam loaded with a series of loads

As shown in figure, we are interested in end shear at A. We need to draw influence line for the support reaction A and a point away from the support at infinitesimal distance on the span for the shear V_A . The influence lines for these cases are shown in Figure 1.6 and 1.7.

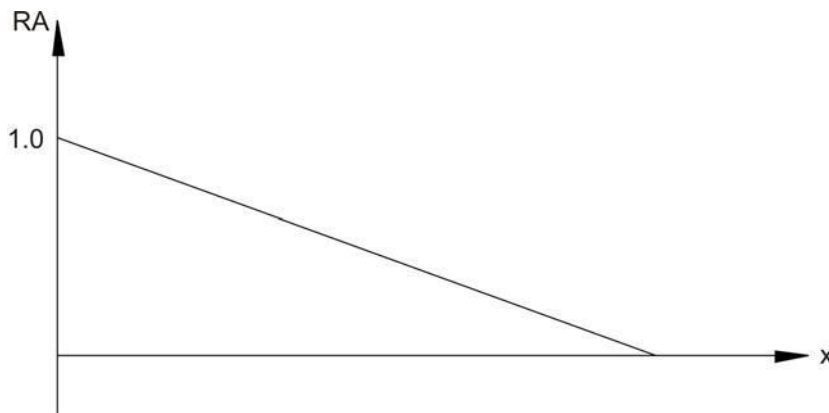


Figure 1.6: Influence line for reaction at support A

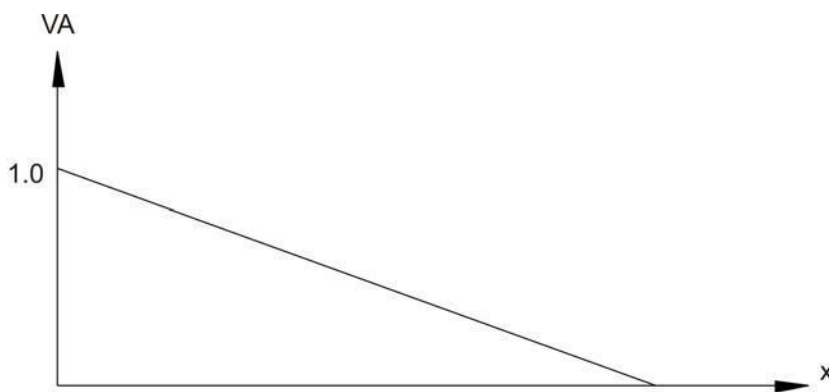


Figure 1.7: Influence line for shear near to support A.

When loads are moving from B to A then as they move closer to A, the shear value will increase. When load passes the support, there could be increase or decrease in shear value depending upon the next point load approaching support A. Using this simple logical approach, we will find out the change in shear value near support and monitor this change from positive value to negative value. Here for the present case let us assume that ΣP is summation of the loads remaining on the beam. When load P_1 crosses support A, then P_2 will approach A. In that case, change in shear will be expressed as

$$dV = \Sigma P_x - P_1$$

When load P_2 crosses support A, then P_3 will approach A. In that case change in shear will be expressed as

$$dV = \Sigma P_y - P_2$$

In case if dV is positive then shear at A has increased and if dV is negative, then shear at A has decreased. Therefore, first load, which crosses and induces negative changes in shear, should be placed on support A.

Numerical Example

Compute maximum end shear for the given beam loaded with moving loads as shown in Figure 1.8.

When first load of 4 kN crosses support A and second load 8 kN is approaching support A, then change in shear can be given by

$$dV = \sum (8 \pm 4)2 - 4 = 0 \quad 10$$

When second load of 8 kN crosses support A and third load 8 kN is approaching support A, then change in shear can be given by

$$dV = \sum (8 \pm 4)3 - 8 = -3.8 \quad 10$$

Hence, as discussed earlier, the second load 8 kN has to be placed on support A to find out maximum end shear (refer Figure 1.9).

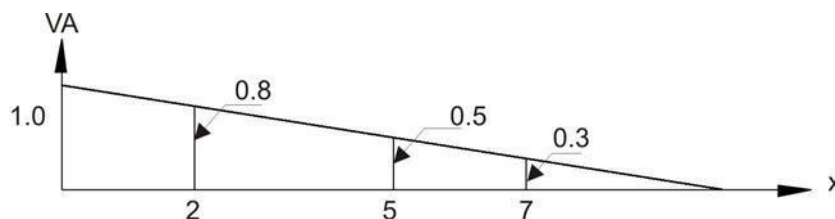


Figure 1.9: Influence line for shear at A.

$$V_A = 4 \times 1 + 8 \times 0.8 + 8 \times 0.5 + 4 \times 0.3 = 15.6 \text{ kN}$$

Maximum shear at a section in a beam supporting a series of moving concentrated loads

In this section we will discuss about the case, when a series of concentrated loads are moving on beam and we are interested to find maximum shear at a section. Let us assume that series of loads are moving from right end to left end as shown in Figure. 1.10.



Figure 1.10: Beam loaded with a series of loads

Monitor the sign of change in shear at the section from positive to negative and apply the concept discussed in earlier section. Following numerical example explains the same.

Numerical Example

The beam is loaded with concentrated loads, which are moving from right to left as shown in Figure 1.12. Compute the maximum shear at the section C.

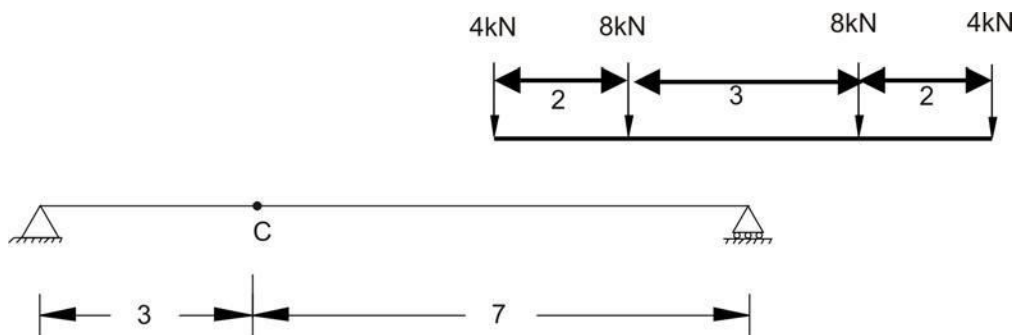


Figure 1.12: Beam loaded with a series of loads

The influence line at section C is shown in following Figure 1.13.

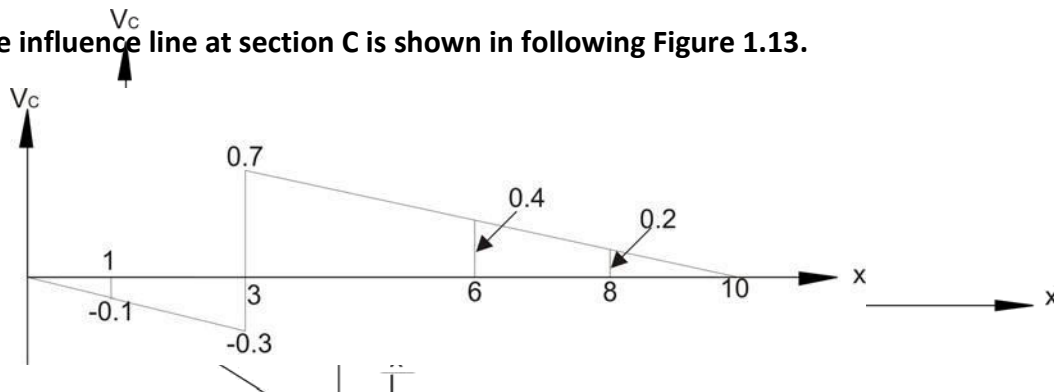


Figure 1.13: Influence line for shear at section C

When first load 4kN crosses section C and second load approaches section C then change in shear at a section can be given by

$$dV = {}^{20}_{10}x^2 - 4 = 0 \quad \text{—————}$$

When second load 8 kN crosses section C and third load approaches section C then change in shear at section can be given by

$$dV = {}^{12}_{10}x^3 - 8 = -4.4 \quad \text{—————}$$

Hence place the second concentrated load at the section and computed shear at a section is given by

$$V_C = 0.1 \times 4 + 0.7 \times 8 + 0.4 \times 8 + 0.2 \times 4 = 9.2 \text{ kN}$$

Maximum Moment at a section in a beam supporting a series of moving concentrated loads

The approach that we discussed earlier can be applied in the present context also to determine the maximum positive moment for the beam supporting a series of moving concentrated loads. The change in moment for a load P_1 that moves from position x_1 to x_2 over a beam can be obtained by multiplying P_1 by the change in ordinate of the influence line i.e. $(y_2 - y_1)$. Let us assume the slope of the influence line (Figure 1.14) is S , then $(y_2 - y_1) = S (x_2 - x_1)$.

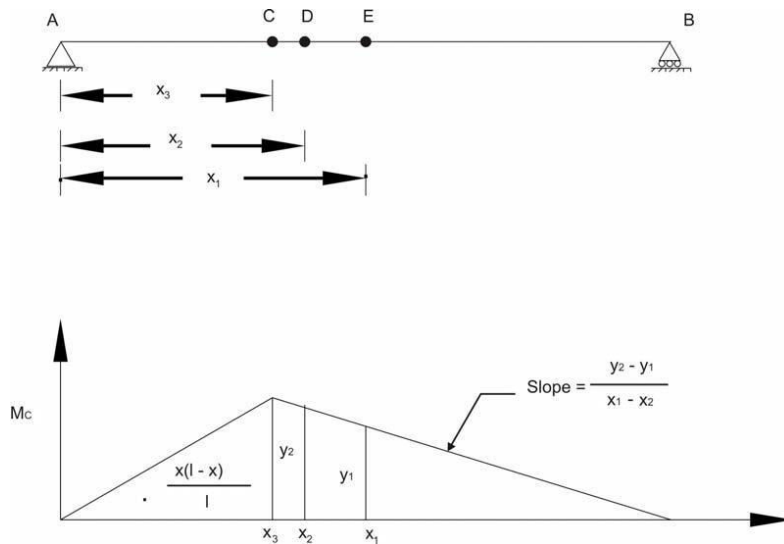


Figure 1.14: Beam and Influence line for moment at section C

Hence change in moment can be given by $dM = P_1 S (x_2 - x_1)$

Let us consider the numerical example for better understanding of the developed concept.