RGPV SOLUTION CS-3001-MATHEMATICS-3-DEC-2017

1. (a) Prove that $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n \frac{\cos nx}{n^2}, -\pi < x < \pi$, Hence show that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$

Solution : Given : $f(x) = x^2, -\pi < x < \pi$ ………(1)

Here, $2L = \pi - (\pi) i.e. 2L = 2\pi \Rightarrow L = \pi$

Suppose the Fourier series of $f(x)$ with period $2L$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$= \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$ [Since $L = \pi$] ………(2)

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 dx$

$$= \quad = 2\int_0^{\pi} x^2 dx$$ [Since $x^2 =$ Even]

$$= \quad a_0 = 2\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{3}\left[\pi^3 - 0\right] = \frac{2\pi^2}{3}$$

and $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\, dx$ [$x\cos nx = odd$]

$$= \quad = 2\int_0^{\pi} x^2 \cos nx\, dx$$

$$= \quad = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \quad a_n = \frac{2}{\pi}\left[\left\{0 + \frac{2\pi(-1)^n}{n^2} - 0\right\} - \{0 - 0 - 0\right] = \frac{4(-1)^n}{n^2}$$

and $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \frac{1}{x}\int_{-\pi}^{\pi} x^2 \sin nx dx$

$$= \quad = 0$$ [$x^2 \sin nx =$ odd]

Putting in equation (1), we get

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos nx$$

$$= \quad f(x) = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - ......\right]$$ ………(3) **Proved**

Since $x = \pi$ is point of discontinuity, then

$$f(\pi) = \frac{1}{2}\left[f(\pi + 0) + f(\pi - 0)\right]$$

$=$ $\quad = \frac{1}{2}[f(\pi+0-2\pi)+f(\pi-0)]$ [Since 2π is period of function]

$=$ $\quad = \frac{1}{2}[f(-\pi+0)+f(\pi-0)]$

$=$ $\quad f(\pi) = \frac{1}{2}\left[(-\pi)^2+\pi^2\right] = \pi^2$ [From (1)]

Putting $x=\pi$ in equation (3), we get

$$f(\pi) = \frac{\pi^2}{3} - 4\left[\frac{\cos\pi}{1^2} - \frac{\cos 2\pi}{2^2} + \frac{\cos 3\pi}{3^2} - .....\right]$$

$=$ $$\pi^2 = \frac{\pi^2}{3} - 4\left[-\frac{1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - ......\right]$$

$=$ $$\pi^2 - \frac{\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...\right]$$

$=$ $$\frac{2x^2}{3\times 4} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + ...$$

$=$ $$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$ **Proved**

(b) Obtain half-range sine series for e^x in < x < 1

Solution : Given : $f(x) = e^x;\ 0 < x < 1$

Here, L = 1

Suppose the half range sine series of $f(x)$ is,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$=$ $$f(x) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$ [Since L = 1] ……..(2)

Now, $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{1}\int_0^1 e^x \sin(n\pi x)dx$

$=$ $$= 2\left[\frac{e^x}{n^2\pi^2+1}[1\sin(n\pi x) - n\pi\cos(n\pi x)]\right]_0^1$$

$=$ $$= \frac{2}{n^2\pi^2+1}\left[\{e^1[\sin(n\pi) - n\pi\cos(n\pi)]\} - \{1[\sin(0) - n\pi\cos(0)]\}\right]$$

$=$ $$= \frac{2}{n^2\pi^2+1}\left[\{e[0+n\pi]\} - \{0-n\pi\}\right]$$

$=$ $$= \frac{2n\pi}{n^2\pi^2+1}[e+1]$$

$\therefore \qquad b_n = \frac{2n\pi}{n^2\pi^2+1}[e+1]$

Putting in equation (2), we get

$$f(x) = 2\pi[e+1]\sum_{n=1}^{\infty}\frac{n\sin(n\pi x)}{n^2\pi^2+1} \qquad \text{Answer}$$

2. (a) Find the Fourier transform of $f(x) = \begin{cases} 1-x^2 & ;|x|\le 1 \\ 0 & ;|x|>1 \end{cases}$. Hence evaluate

$\int_0^{\infty}\frac{x\cos x - \sin x}{x^3}\cos\left(\frac{x}{2}\right)dx$

Solution: Given the function: $f(x) = \begin{cases} 1-x^2 & ;|x|\le 1 \\ 0 & ;|x|>1 \end{cases}$

The Fourier transform of a function $F(x)$ is given by

$$f(p) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}F(x)e^{jpx}dx \qquad \ldots\ldots\ldots(1)$$

Substituting the values of $f(x)$ in (1), we get

$$f(p) = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{jpx}dx$$

$$= \qquad f(p) = \left[(1-x^2)\left(\frac{e^{ipx}}{ip}\right) - (2x)\left(\frac{e^{ipx}}{(ip)^2}\right) + (-2)\left(\frac{e^{ipx}}{(ip)^3}\right)\right]_{-1}^{1}$$

$$= \qquad f(p) = \frac{1}{\sqrt{2\pi}}\left[\left\{0 - 2\left(\frac{e^{ip}}{p^2}\right) + 2\left(\frac{e^{ip}}{ip^3}\right)\right\} - \left\{0 + 2\left(\frac{e^{-ip}}{p^2}\right) + 2\left(\frac{e^{-ip}}{ip^3}\right)\right\}\right]$$

$$= \qquad f(p) = \frac{1}{\sqrt{2\pi}}\left[-2\left(\frac{e^{ip}+e^{-ip}}{p^2}\right) + 2\left(\frac{e^{ip}-e^{-ip}}{ip^3}\right)\right]$$

$$= \qquad f(p) = \frac{1}{\sqrt{2\pi}}\left[-\frac{2}{p^2}(2\cos p) + \frac{2}{ip^3}(2i\sin p)\right]\left[\because \sin p = \frac{e^{ip}-e^{-ip}}{2i}, \cos p = \frac{e^{ip}+e^{-ip}}{2}\right]$$

$$= \qquad f(p) = -\frac{4}{p^3\sqrt{2\pi}}[p\cos p - \sin p] \qquad \ldots\ldots\ldots\ldots(2)$$

Since inverse Fourier transform of $f(p)$ is,

$$\therefore \qquad F(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left\{-\frac{4}{p^3\sqrt{2\pi}}[p\cos p - \sin p]\right\}e^{-ipx}dx$$

$$= \qquad -\frac{2}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{p^3}(p\cos p - \sin p)\right]e^{-ipx}dp = F(x) \qquad \ldots\ldots\ldots(3)$$

Since $x = \frac{1}{2}$, is point of continuity, then $F\left(\frac{1}{2}\right) = 1 - \left(\frac{1}{2}\right)^2 = \frac{3}{4}$

Putting in $x = \frac{1}{2}$ in equation (3), we get

$$-\frac{2}{\pi}\int_{-\infty}^{\infty}\left[\frac{1}{p^3}(p\cos p - \sin p)\right]e^{-ip/2}dp = \frac{3}{4}$$

$$= \qquad \int_{-\infty}^{\infty}\left[\frac{1}{p^3}(p\cos p - \sin p)\right]\left(\cos\frac{p}{2} - i\sin\frac{p}{2}\right)dp = -\frac{3\pi}{8}$$

Comparing real and imaginary part, we get

$$\int_{-\infty}^{\infty}\left[\frac{1}{p^3}(p\cos p - \sin p)\right]\cos\frac{p}{2}\,dp = -\frac{3\pi}{8}$$

$$= \qquad 2\int_{0}^{\infty}\left(\frac{p\cos p - \sin p}{p^3}\right)\cos\frac{p}{2}\,dp = -\frac{3\pi}{8} \qquad \text{[Since the function is even]}$$

$$= \qquad \int_{0}^{\infty}\left(\frac{p\cos p - \sin p}{p^3}\right)\cos\frac{p}{2}\,dp = -\frac{3\pi}{16}$$

$$\therefore \qquad \int_{0}^{\infty}\left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx = -\frac{3\pi}{16} \qquad \text{[By definite integral property]}$$

(b) Using Laplace Transform to solve the differential equation

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t; \textbf{ When } x = 2, \frac{dx}{dt} = -1\, at\, t = 0$$

Solution : Given the differential equation is,

$$x''(t) - 2x'(t) + x(t) = e^t \qquad \ldots\ldots\ldots\ldots(1)$$

With initial condition are: $x(0) = 2$ and $x'(0) = -1$

Taking Laplace transform of (1) on both sides, we get

$$L\{x''(t)\} - 2L\{x'(t)\} + L\{x(t)\} = L\{e^t\}$$

$$= \qquad \left[p^2x(p) - px(0) - x'(0)\right] + 2L\{px(p) - x(0)\} + x(p) = \frac{1}{p-1}$$

Putting the initial values, $x(0) = 2$ and $x'(0) = -1,$ we ge

$$\therefore \qquad \left[p^2x(p) - 2p + 1\right] - 2\left[px(p) - 2\right] + x(p) = \frac{1}{p-1}$$

$$= \qquad \left(p^2 - 2p + 1\right)x(p) - 2p + 1 + 4 = \frac{1}{p-1}$$

$$= \qquad (p-1)^2 x(p) = \frac{1}{p-1} + 2p - 5$$

$$= \qquad L\{x(t)\} = \frac{1}{(p-1)^3} + \frac{2p-5}{(p-1)^2}$$

$= \quad x(t) = L^{-1}\left\{\frac{1}{(p-1)^3}\right\} + L^{-1}\left\{\frac{2p-5}{(p-1)^2}\right\}$

$= \quad = L^{-1}\left\{\frac{1}{(p-1)^3}\right\} + L^{-1}\left\{\frac{2(p-1)-3}{(p-1)^2}\right\}$

$= \quad x(t) = e^t L^{-1}\left\{\frac{1}{p^3}\right\} + e^t L^{-1}\left\{\frac{2p-3}{p^2}\right\}$

$= \quad x(t) = e^t\left(\frac{t^2}{2}\right) + 2e^t L^{-1}\left\{\frac{2}{p} - \frac{3}{p^2}\right\}$

Thus $\boxed{x(t) = e^t\left(\frac{t^2}{2}\right) + e^t(2-3t)}$ **Answer**

3. (a) Find the Laplace transform of $\frac{1-\cos t}{t^2}$

Solution : Let $F(t) = 1 - \cos t$

Taking Laplace transform on both sides, we get

$$L\{F(t)\} = [L\{1\} - \{\cos t\}]$$

$$= \quad = \frac{1}{2}\left[\frac{1}{p} - \frac{p}{p^2+1}\right] = f(p) \qquad \text{[Say]}$$

By Laplace transform by division of t, we have

$$L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(p)\,dp$$

$$\therefore \quad L\left\{\frac{1-\cos t}{t}\right\} = \int_p^\infty \left[\frac{1}{p} - \frac{p}{p^2+1}\right]dp$$

$$= \quad = \left[\log p - \frac{1}{2}\log(p^2+1)\right]_p^\infty = \frac{1}{2}\left[2\log p - \log(p^2+1)\right]_p^\infty$$

$$= \quad = \frac{1}{2}\left[\log\left(\frac{p^2}{p^2+1}\right)\right]_p^\infty = \frac{1}{2}\left[\log\left(\frac{1}{1+\frac{1}{p^2}}\right)\right]_p^\infty$$

$$= \quad = \frac{1}{2}\left[\log\left(\frac{1}{1+\frac{1}{\infty}}\right) - \log\left(\frac{1}{1+\frac{1}{p^2}}\right)\right] = \frac{1}{2}\left[0 - \log\left(\frac{1}{1+\frac{1}{p^2}}\right)\right]$$

$$= \quad = -\frac{1}{2}\left[\log\left(\frac{p^2}{p^2+1}\right)\right] = \frac{1}{2}\left[\log\left(\frac{p^2+1}{p^2}\right)\right]$$

Thus $$L\left\{\frac{1-\cos t}{t}\right\}=\frac{1}{2}\left[\log\left(\frac{p^2+1}{P^2}\right)\right]=f_1(p)\ \{Say\}$$ ....(1)

Let $F_1(t)=\dfrac{1-\cos t}{t}$ *and* $f_1(p)=\dfrac{1}{2}\log\left(\dfrac{p^2+1}{p^2}\right)$

Applying formula for Laplace transform by division of t, we have

$$L\left\{\frac{F_1(t)}{t}\right\}=\int_p^\infty f_1(p)dp$$

$$\therefore \qquad L\left\{\frac{1-\cos t}{t^2}\right\}=L\left\{\frac{\left(\frac{1-\cos t}{t}\right)}{t}\right\}=\int_p^\infty f(p)dp=\frac{1}{2}\int_p^\infty \log\left(\frac{p^2+1}{p^2}\right)dp$$

$$= \qquad =\frac{1}{2}\int_p^\infty \overset{I}{1}.\left[\log\left(p^2+1\right)^{II}-\log\left(p^2\right)\right]dp$$

Applying integration by parts formula, we get.

$$=\frac{1}{2}\left[p.\left\{\log\left(p^2+1\right)-\log\left(p^2\right)\right\}\right]_p^\infty-\frac{1}{2}\int_p^\infty p.\left(\frac{2p}{p^2+1}-\frac{2}{p}\right)dp$$

$$= \qquad =\frac{1}{2}\left[p.\log\left(\frac{p^2+1}{p^2}\right)\right]_p^\infty+\int_p^\infty\frac{1}{p^2+1}dp$$

$$= \qquad =\frac{1}{2}\left[0-p.\log\left(\frac{p^2+1}{p^2}\right)\right]+\left[\tan^{-1}(p)\right]_p^\infty$$

$$= \qquad =-\frac{p}{2}.\log\left(\frac{p^2+1}{p^2}\right)+\left[\tan^{-1}(\infty)-\tan^{-1}(p)\right]$$

$$= \qquad =-\frac{p}{2}.\log\left(\frac{p^2+1}{p^2}\right)+\left[\frac{\pi}{2}-\tan^{-1}(p)\right]$$

$$= \qquad =-\frac{p}{2}.\log\left(\frac{p^2+1}{p^2}\right)+\cot^{-1}(p)$$

Thus $$\boxed{L\left\{\frac{1-\cos t}{t^2}\right\}=-\frac{p}{2}.\log\left(\frac{p^2+1}{p^2}+\cot^{-1}(p)\right)}$$ **Answer**

(b) Using the Convolution theorem and find $L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}$

Solution : Suppose $f(s)=\frac{s}{s^2+1}$ and $g(s)=\frac{1}{s^2+4}$

$\therefore$ $L^{-1}\{f(s)\}=L^{-1}\left\{\frac{s}{s^2+1}\right\}=\cos t=F(t)$

And $L^{-1}\{g(s)\}=L^{-1}\left\{\frac{1}{s^2+4}\right\}=\frac{1}{2}\sin 2t=G(t)$

By Convolution theorem of Inverse Laplace transform, we have

$$L^{-1}\{f(s).g(s)\}=\int_0^t F(x)G(t-x)dx$$

$\therefore$ $$L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}=\frac{1}{2}\int_0^t \cos x\sin(2t-2x)dx$$

$$=\;=\frac{1}{2}\int_0^t 2\sin(2t-2x)\cos x dx=\frac{1}{4}\int_0^t[\sin(2t-2x+x)+\sin(2t-2x-x)]dx$$

$$[\because 2\sin A\cos B=\sin(A+B)+\sin(A-B)]$$

$$=\;=\frac{1}{2}\int_0^t[\sin(2t-x)+\sin(2t-3x)]dx=\frac{1}{4}\left[\cos(2t-x)+\frac{\cos(2t-3x)}{3}\right]_0^t$$

$$=\;=\frac{1}{4}\left[\left\{\cos t+\frac{\cos t}{3}\right\}-\left\{\cos 2t+\frac{\cos 2t}{3}\right\}\right]=\frac{1}{3}[\cos t-\cos 2t]$$

Thus $$L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}=\frac{1}{3}[\cos t-\cos 2t]$$ **Answer**

4. (a) Define :

(i). Probability density function for continuous random variable.

(ii). Mean and variance of random variables.

Solution :

(i). Probability density function for CRV:

If X is continuous random variable defined in $(-\infty,\infty)$, then the function f(x) is said to p.d.f. if $\int_{-\infty}^{\infty} f(x)dx=1$, When $-\infty<x<\infty$

(ii). Mean and Variance of CRV.

If M is the mean, then $M=\int_{-\infty}^{\infty} xf(x)dx$, When $-\infty<x<\infty$

The $Variance(V)=\mu'_2-(\mu_1')^2$ where $\mu_2'=\int_{-\infty}^{\infty} x^2 f(x)dx;\ When -\infty<x<\infty$ and

$\mu_1'=\int_{-\infty}^{\infty} xf(x)dx, When -\infty<x<\infty$

(b) Find the mean and variance for Binomial distribution.

Solution : ***(i). Mean of Binomial Distribution:***

We know that by binomial distribution

$$P(X=r)={}^{n}C_{r}p^{r}q^{n-r}$$

Formula for mean of B.D. is,

$$m=\sum_{r=0}^{n} r.P(X=r)$$

$\therefore$ $$=\sum_{r=0}^{n} r.{}^{n}C_{r}p^{r}q^{n-r}=\sum_{r=1}^{n} r.{}^{n}C_{r}p^{r}q^{n-r} \qquad [\because \text{first term is zero}]$$

$=$ $$\sum_{r=1}^{n} n.{}^{n-1}C_{r-1}p^{r}q^{n-r} \qquad [\because r.{}^{n}C_{r}=n.{}^{n-1}C_{r-1}]$$

$=$ $$=n\,p\sum_{r=1}^{n} {}^{n-1}C_{r-1}p^{r-1}q^{(n-1)-(r-1)}$$

$=$ $$=n\,p(q+p)^{n-1}=n\,p \qquad [\because q+p=1]$$

Hence, $m=n\,p$

(ii). *Variance of Binomial Distribution:*

We know that by binomial distribution

$$P(X=r)={}^{n}C_{r}p^{r}q^{n-r}$$

Formula for variance of B.D. is,

$$V=\sum_{r=0}^{n} r^{2}.P(X=r)-(mean)^{2}$$

$\therefore$ $$=\sum_{r=0}^{n}\left[r+r(r-1\right].{}^{n}C_{r}p^{r}q^{n-r}-m^{2}$$

$=$ $$=\sum_{r=0}^{n} r.{}^{n}C_{r}p^{r}q^{n-r}+\sum_{r=0}^{n} r.(r-1){}^{n}C_{r}p^{r}q^{n-r}-n^{2}p^{2}$$

$=$ $$=np+\sum_{r=2}^{n} r.(r-1)^{n}C_{r}r^{r}q^{n-r}-n^{2}p^{2} \qquad [\because \text{ First two terms are zero}]$$

$=$ $$=np+\sum_{r=2}^{n} n.(n-1)\,{}^{n-2}C_{r-2}p^{2}q^{n-r}-n^{2}p^{2}\left[\because r.(r-1)\,{}^{n}C_{r}=n.(n-1)\,{}^{n-2}C_{r-2}\right]$$

$=$ $$=np+n.(n-1)p^{2}\sum_{r=2}^{n} {}^{n-2}C_{r-2}p^{r-2}q^{(n-2)-(r-2)}-n^{2}p^{2}$$

$=$ $$=np+\left(n^{2}p^{2}-np^{2}\right)(q+p)^{n-2}-n^{2}p^{2}$$

$=$ $$=np+\cancel{n^{2}p^{2}}-np^{2}-\cancel{n^{2}p^{2}} \qquad [\because q+p=1]$$

$=$ $$np\,(1-p)=npq$$

Hence, $V = n\,p\,q$

5. (a) Fit Poisson's distribution to the following and calculate the theoretical frequencies $e^{-5} = 0.61$

Death	:	0	1	2	3	4
Frequency	:	122	60	15	2	1

Solution : Given, $n = 5$ and $N = \sum f = 200$

The expected frequency of Poisson distribution

$$f_e = N\,P(X = r) = 200\left[\frac{e^{-m} m^r}{\lfloor 2}\right] \qquad \ldots\ldots\ldots(1)$$

Mean $m = \dfrac{\sum f\,r}{N} = \dfrac{100}{200} = 0.5$

Expected frequency distribution table

r	f	$f.r$	$f_e = 122\times\left[\frac{(0.5)^r}{\lfloor r}\right]$
0	122	0	122
1	60	60	61
2	15	30	15.25~15
3	2	6	2.541~3
4	1	4	0.3177~0
Total	**200**	$\sum f.r = 100$	

Putting in equation (1), we get

$$f_e = 200\left[\frac{e^{-0.5}(0.5)^r}{\lfloor r}\right] = 200\times 0.61\left[\frac{(0.5)^r}{\lfloor r}\right]$$

$$= \qquad f_e = 122\times\left[\frac{(0.5)^r}{\lfloor r}\right]$$

Putting r = 0, 1, 2, 3, 4, we get the expected frequency are 122, 61, 15, 3 and 0 respectively.

(b) Show that the mean deviation from mean of the normal distribution is 4/5 times of standard deviation.

Solution: We know that by the definition of normal distribution function is,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-m)^2}{2\sigma^2}} \; for -\infty < x < \infty \qquad \ldots\ldots\ldots\ldots(1)$$

We know that the formula of mean deviation about mean of normal distribution is,

Mean deviation = $\int_{-\infty}^{\infty} |x - m|\, f(x)\, dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} |x-m|^{-\frac{(x-m)^2}{2\sigma^2}} dx \qquad \ldots\ldots\ldots\ldots(2)$$

Putting, $z = \frac{x-m}{\sigma} \Rightarrow x - m = z\sigma \; i.e. dx = \sigma\, dz$

$\therefore$ From (2) $Mean\ deviation = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\, z\sigma \,| e^{-\frac{z^2}{2}} dz$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |\, z \,| e^{-\frac{z^2}{2}} dz$$

$$M.D. = \frac{2\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z\, e^{-\frac{z^2}{2}} dz \qquad \ldots\ldots\ldots(3)$$ [$\because$ |z| $e^{-z^2/2}$ is even function]

Putting, $t = \frac{z^2}{2} \Rightarrow 2t = z^2$ so that $z\,dz = dt$

$\therefore$ From (3), $Mean\ deviation = \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t} dt$

$$= \frac{2\sigma}{\sqrt{2\pi}} \left[-e^{-t}\right]_0^{\infty} = \frac{2\sigma}{\sqrt{2\pi}} \left[-e^{-\infty} + e^{-0}\right]_0^{\infty} = \frac{2\sigma}{\sqrt{2\pi}} [0+1]$$

$$= \left(\frac{\sqrt{2}}{\sqrt{\pi}}\right)\sigma$$

$$\text{M.D.} = 0.8\ \sigma = \left(\frac{4}{5}\right)\sigma$$

Thus, $M.D. = \left(\frac{4}{5}\right)\sigma$ **Hence Proved**

6. (a) By the method of least squares. Find the straight line that best fits the following data:

x	**:**	**1**	**2**	**3**	**4**	**5**
y	**:**	**14**	**27**	**40**	**55**	**68**

Solution: Suppose straight line y as dependent and x as a independent variable is

$$y = a + bx \qquad \ldots\ldots\ldots\ldots\ (1)$$

Here two unknown constants, then the two normal equations are,

$$\sum y = ma + b\sum x \qquad \ldots\ldots\ldots\ldots(2)$$

and $\sum xy = a\sum x + b\sum x^2 \qquad \ldots\ldots\ldots\ldots(3)$

Table :

x	y	x.y	x^2
1	14	14	1
2	27	54	4

3	40	120	9
4	55	220	16
5	68	340	25
$\sum x = 15$	$\sum y = 204$	$\sum x.y = 748$	$\sum x^2 = 55$

Here, m = 5

Putting in equation (2) and (3), we get

$$5a + 15b = 204 \qquad \ldots\ldots(4)$$

and $$15a + 55b = 748$$

Solving equation (4) and (5), we get

$$a = 0 \text{ and } b = 13.6$$

Putting in equation (1), we get

$$y = 13.6x$$ **Answer**

(b) The profit of a certain company in the x^{th} year of its life are given by:

x	:	1	2	3	4	5
y	:	1250	1400	1650	1950	2300

Taking u = x – 3 and 50 v = y – 1650, show that the parabola of second degree of y on x is:

$$y = 1140.05 + 72.1x + 32.15x^2$$

Solution : Given the new variable u and v such that

$$u = x - 3 \, and \, v = \frac{y - 1650}{50}$$

Suppose the second degree parabola equation with variables u and v is,

$$v = a + bu + cu^2 \qquad \ldots\ldots\ldots..(1)$$

$\therefore$ The normal equations are,

$$\sum v = ma + b\sum u + c\sum u^2 \qquad \ldots\ldots\ldots\ldots(2)$$

$$\sum uv = a\sum u + b\sum u^2 + c\sum u^3 \qquad \ldots\ldots\ldots\ldots(3)$$

and $$\sum u^2 v = a\sum u^2 + b\sum u^3 + c\sum u^4 \qquad \ldots\ldots\ldots\ldots.(4)$$

Table for fitting of curve as follows,

x	y	u = x – 3	$v = \frac{y-1650}{50}$	u.v	u^2	u^2v	u^3	u^4
1	1250	-2	-8	16	4	-32	-8	16
2	1400	-1	-5	5	1	-5	-1	1
3	1650	0	0	0	0	0	0	0
4	1950	1	6	6	1	6	1	1
5	2300	2	13	26	4	52	8	16
15	**8550**	**0**	**6**	**53**	**10**	**21**	**0**	**34**

Now we have

$$m = 5, \sum u = 0, \sum v = 6, \sum uv = 53, \sum u^2 = 10, \sum u^3 = 0, \sum u^4 = 34 and \sum u^2 v = 21$$

Putting these values in equation (2), (3) and (4), we get

$5a + b(0) + 10c + 6 \Rightarrow 5a + 10c = 6$ ……….. (5)

$a(0) + 10b + c(0) = 53 \Rightarrow \quad 10b = 53 i.e. b = 5.3$ ………..(6)

$10a + b(0) + 34c = 21 \Rightarrow \quad 10a + 34c = 21$ ………..(7)

On solving equations (5) and (7), we get

$a = -0.085714$ and $c = 0.642857$

Putting the values of a, b and c in equation (1), we get

$v = -0.085714 + 5.3u + 0.642857u^2$

$= \quad \frac{y-1650}{50} = -0.085714 + 5.3(x-3) + 0.642857(x-3)^2$

$= \quad = -0.085714 + 5.3x - 15.9 + 0.642857(x^2 - 6x + 9)$

$= \quad = -0.085714 + 5.3x - 15.9 + 0.642857x^2 - 3.857142x + 5.785713$

$= \quad = 0.642857x^2 + 1.442858x - 10.200001$

$= \quad y - 1650 = 32.14285x^2 + 72.1429x - 510.00005$

$= \quad y = 1139.99995 + 72.1429x + 32.14285x^2$

$\therefore \quad y = 1139.99995 + 72.1429x + 32.14285x^2$ **Hence Proved**

7. (a) Find a Fourier Series for $f(x) = \begin{cases} -\pi; -\pi < x < 0 \\ x; \; x < x < \pi \end{cases}$ and deduce that;

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

Solution : Given : $f(x)=\begin{cases} -\pi; -\pi<x<0 \\ x;\ \ x<x<\pi \end{cases}$

Here, $2L=\pi-(-\pi) i.e. 2L=2\pi \Rightarrow L=\pi$

Suppose the Fourier series of $f(x)$ with period $2L$ is,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)+\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$=\qquad f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx+\sum_{n=1}^{\infty} b_n \sin nx \qquad$ [Since $L=\pi$] ……(2)

Now, $\quad a_0=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx$

$=\qquad =\frac{1}{\pi}\int_{-\pi}^{0} -\pi\, dx+\frac{1}{\pi}\int_{0}^{\pi} x\, dx$

$=\qquad =-1[x]_{-x}^{0}+\frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}=-[0+\pi]+\frac{1}{2\pi}\left[\pi^2-0\right]$

$=\qquad =-\pi+\frac{\pi}{2}=-\frac{\pi}{2}$

$\therefore \qquad a_0=-\frac{\pi}{2}$

Now, $\quad a_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-\pi)\cos nx dx+\frac{1}{\pi}\int_{0}^{\pi} x\cos nx dx$$

$=\qquad =-\left[\frac{\sin nx}{n}\right]_{-\pi}^{0}+\frac{1}{\pi}\left[\left(x\frac{\sin nx}{n}\right)-1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$

$=\qquad =-\frac{1}{n}[0-0]+\frac{1}{\pi}\left[\left\{0+\frac{(-1)^n}{n^2}\right\}-\left\{0+\frac{1}{n^2}\right\}\right]$

$\therefore \qquad a_n=\frac{1}{n^2\pi}\left[(-1)^n-1\right]$

and $\quad b_n=\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$

$$=\frac{1}{\pi}\int_{-\pi}^{0}(-\pi)\sin nx dx+\frac{1}{\pi}\int_{0}^{\pi} x\sin nx dx$$

$$=-\left[-\frac{\cos nx}{n}\right]_{-\pi}^{0}+\frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)-1\left(\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{n}\left[1-(-1)^n\right]+\frac{1}{\pi}\left[\left\{-x\frac{(-1)^n}{n}-0\right\}-\{0+0\right] = \frac{1}{\pi}\left[1-(-1)^n\right]-\frac{(-1)^n}{n}$$

$$\boxed{b_n = \frac{1}{n}\left[1-2(-1)^n\right]}$$

Putting in equation (1), we get

$$f(x) = -\frac{\pi}{4}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\left[(-1)^n-1\right]}{n^2}\cos nx+\sum_{n=1}^{\infty}\frac{1}{n}\left[1-2(-1)^n\right]\sin nx$$

$$= \qquad f(x) = -\frac{\pi}{4}-\frac{2}{\pi}\left[\frac{\cos x}{1^2}+\frac{\cos 3x}{3^2}+\frac{\cos 5x}{5^2}+.....\right]+\left[\frac{3\sin x}{1}-\frac{\sin 2x}{2}+\frac{3\sin 3x}{3}-\frac{\sin 4x}{4}+...\right] \quad ....(2)$$

Since $x=0$ is point of discontinuity, then

$$f(0) = \frac{1}{2}\left[f(0+0)+f(0-0)\right]$$

$$= \qquad f(0) = \frac{1}{2}\left[0-\pi\right] = -\frac{\pi}{2} \qquad \text{[From (1)]}$$

Putting in equation (2), we get

$$f(0) = -\frac{\pi}{4}-\frac{2}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....\right]+0$$

$$= \qquad -\frac{\pi}{2}+\frac{\pi}{4} = -\frac{2}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....\right]$$

$$= \qquad -\frac{\pi}{4} = -\frac{2}{\pi}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....\right]$$

$$\therefore \qquad \frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+....$$ **Hence Proved**

(b) Find the Fourier cosine transform of $f(x)=\dfrac{1}{1+x^2}$ and hence find Fourier sine transform of

$$F(x) = \frac{1}{1+x^2}$$

Solution: Suppose $F(x)=\dfrac{1}{1+x^2}$

The Fourier cosine transform of F(x) is,

$$f_C(p) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} F(x)\cos px\,dx$$

$$\therefore \qquad f_C(p) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{1}{1+x^2}\cos px\,dx = 1 \ [Say] \qquad ....(1)$$

Differentiating w.r.t., p, we get

$$\frac{d}{dp}I = \frac{d}{dp}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+x^2}\cos px\,dx$$

$$= \qquad = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+x^2}\frac{\partial}{\partial p}(\cos px)dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x}{1+x^2}\sin px\,dx$$

$$= \qquad \frac{dI}{dp} = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x^2}{x(1+x^2)}\sin px dx = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{(1+x^2-1)}{x(1+x^2)}\ \sin px dx \qquad \text{[Adding and subtract 1]}$$

$$= \qquad \frac{dI}{dp} = -\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin px}{x}dx + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin px}{x(1+x^2)}dx$$

$$= \qquad \frac{dI}{dp} = -\sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right) + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin px}{x(1+x^2)}dx \qquad \left[\because \int_0^\infty \frac{\sin ax}{x}dx = \frac{\pi}{2}\right]$$

$$= \qquad \frac{dI}{dp} = -\sqrt{\frac{\pi}{2}} + \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\sin px}{x(1+x^2)}dx \qquad \text{……….(2)}$$

Again differentiating w.r.t., p, we get

$$\frac{d^2I}{dp^2} = 0 + \frac{d}{dp}\sqrt{\frac{2}{\pi}}\ \int_0^\infty \frac{\sin px}{x(1+x^2)}dx \qquad \text{From (1)}$$

$$= \qquad \frac{d^2I}{dp^2} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{x\cos px}{x(1+x^2)}dx = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\cos px}{1+x^2}dx = I$$

$$= \qquad \frac{d^2I}{dp^2} - 1 = 0 \qquad \text{……….. (3)}$$

This is Linear differential equation of higher order.

$\therefore$ The solution of (3) is,

$$I = c_1e^p + c_2e^{-p} \qquad \text{…………..(1)}$$

Differentiating w.r.t., p, we get

$$\frac{dI}{dp} = c_1e^p - c_2e^p \qquad \text{…………(2)}$$

Putting $p = 0,$ in equation (1) and (4) we get

$$I = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+x^2}dx$$

$$= \frac{\sqrt{2}}{\pi}\left[\tan^{-1}x\right]_0^\infty = \sqrt{\frac{2}{\pi}}\left(\frac{\pi}{2}\right) = \sqrt{\frac{\pi}{2}}$$

and $\quad c_1 + c_2 = I \Rightarrow c_1 + c_2 = \sqrt{\frac{\pi}{2}}$ ……….(6)

Again Putting $p = 0,$ in equation (2) and (5) we get

$$\frac{dI}{dp} = -\sqrt{\frac{\pi}{2}} + 0 \Rightarrow \frac{dI}{dp} = -\sqrt{\frac{\pi}{2}} \text{ and } c_1 - c_2 = -\sqrt{\frac{\pi}{2}} \qquad \ldots\ldots\ldots.(7)$$

Solve (6) and (7), we get

$$c_1 = 0 \text{ and } c_2 = \sqrt{\frac{\pi}{2}}$$

$\therefore$ From (4), we get

$$I = \sqrt{\frac{\pi}{2}} e^{-p}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\cos px}{1+x^2} dx = \sqrt{\frac{\pi}{2}} e^{-p} \text{ i.e., } \boxed{F_C\left\{\frac{1}{1+x^2}\right\} = \frac{\sqrt{\pi}}{2} e^{-p}}$$

Answer

Differentiating w.r.t., p, we get

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{-x \sin px}{1+x^2} dx = -\sqrt{\frac{\pi}{2}} e^{-p}$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{x \sin px}{1+x^2} dx = \sqrt{\frac{\pi}{2}} e^{-p} \Rightarrow \boxed{F_S\left\{\frac{1}{1+x^2}\right\} = \frac{\sqrt{\pi}}{2} e^{-p}}$$

Answer

8. (a) For a Poisson distribution with mean m, show that $\mu_{r+1} = m\mu_{e-1} + m\frac{d\mu_r}{dm}$

Where $\mu_r = \sum_{x=0}^{\infty} (x-m)^r \frac{e^{-m} m^x}{\lfloor x}$

Solution : Given $\mu_r = \sum_{x=0}^{\infty} (x-m)^r \frac{e^{-m} m^x}{\lfloor x} \qquad \ldots\ldots\ldots\ldots(1)$

Now, $\frac{d\mu_r}{dm} = \frac{d}{dm}\left[\sum_{x=0}^{\infty} (x-m)^r \frac{e^{-m} m^x}{\lfloor x}\right]$

$$= -\sum_{x=0}^{\infty} (x-m)^r \frac{e^{-m} m^x}{\lfloor x} + = -\sum_{x=0}^{\infty} \frac{(x-m)^r}{\lfloor x}\left[x m^{x-1} e^{-m} - m^x e^{-m}\right]$$

$$= -r\mu_{r-1} + e^{-m} \sum_{x=0}^{\infty} \frac{(x-m)^r}{\lfloor x} m^{x-1}[x-m] \qquad \text{[From (1)]}$$

$$\frac{d\mu_r}{dm} = -r\mu_{r-1} + \frac{1}{m} \sum_{x=0}^{\infty} \frac{(x-m)^{r+1}}{\lfloor x} e^{-m} m^x$$

$$\frac{d\mu_r}{dm} = -r\mu_{r-1} + \frac{1}{m}\mu_{r+1} \qquad \text{[From (1)]}$$

$$= \boxed{m\frac{d\mu_r}{dm} + mr\mu_{r-1} = \mu_{r+1}}$$

(b) Evaluate by using Laplace transform

(i). $\int_0^\infty t e^{-4t} \sin t \, dt$ (ii). $\int_0^\infty \frac{e^{-t} \sin t}{t} dt$

Solution : (i). $L\{\sin t\} = \frac{1}{p^2+1} = f(p)$

By Multiplication property, we have

$$L\{t \sin t\} = (-1)\frac{d}{dp} f(p)$$

$$= \quad L\{t \sin t\} = -\frac{d}{dp}\left(\frac{1}{p^2+1}\right) = \frac{2p}{(p^2+1)^2}$$

$$= \quad \int_0^\infty e^{-pt}(t \sin t) dt = \frac{2p}{(p^2+1)^2}$$ [By Definition of L.T.]

Putting p = 4, we get

$$\int_0^\infty e^{-pt}(t \sin t) dt = \frac{2(4)}{(4^2+1)^2} = \frac{8}{289}$$ **Answer**

(ii). Let $F(t) = \sin t$

$$\therefore \quad L\{F(t)\} = L\{\sin t\} = \frac{1}{p^2+1} = f(p)$$

By Laplace transform of division of t, we have

$$L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(p) \, dp \qquad \text{........(1)}$$

$$\therefore \quad L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty \frac{1}{p^2+1} dp = \left[\tan^{-1} p\right]_p^\infty$$

$$= \quad = \tan^{-1}(\infty) - \tan^{-1}(p)$$

$$= \quad L\left(\frac{\sin t}{t}\right) = \frac{\pi}{2} - \tan^{-1}(p)$$

$$= \quad \int_0^\infty e^{-pt}\left(\frac{\sin t}{t}\right) dt = \frac{\pi}{2} - \tan^{-1}(p)$$ [By Definition of L.T.]

Putting p =1, we get

$$\int_0^\infty e^{-t}\left(\frac{\sin t}{t}\right) dt = \frac{\pi}{2} - \tan^{-1}(1)$$

$$\int_0^\infty e^{-t}\left(\frac{\sin t}{t}\right) dt = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

$$\therefore \quad \boxed{\int_0^\infty e^{-t}\left(\frac{\sin t}{t}\right) dt = \frac{\pi}{4}}$$ **Answer**