RGPV BT-2002 MATHEMATICS-2 JUN 2018 SOLUTION

1. a) Find the rank of matrix $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Solution : Now will find the rank of matrix by Echelon form

Give $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Applying, $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

Applying, $R_3 \to R_3 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

Clearly number of non-zero two rows, then

$$\rho(A) = 2$$

b) Find the solution of system of equations

$$2x + 3y + 4z = 11$$

$$x + 5y + 7z = 15$$

$$3x + 11y + 13z = 25$$

Solution : Give the system of equation is

$$\begin{bmatrix} 2 & 3 & 4 \\ 1 & 5 & 7 \\ 3 & 11 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \\ 25 \end{bmatrix}$$

$\Rightarrow$ AX = B

This is Non-Homogeneous Linear system of equation, and then augmented matrix is

$$C = [A:B] = \left[\begin{array}{ccc|c} 2 & 3 & 4 & 11 \\ 1 & 5 & 7 & 15 \\ 3 & 11 & 13 & 25 \end{array}\right]$$

Applying, $R_1 \leftrightarrow R_2$

$$C \sim \left[\begin{array}{ccc|c} 1 & 5 & 7 & 15 \\ 2 & 3 & 4 & 11 \\ 3 & 11 & 13 & 25 \end{array}\right]$$

Applying, $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$C \sim \left[\begin{array}{ccc|c} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & -4 & -8 & -20 \end{array}\right]$$

Applying, $R_3 \to R_3 / (-4)$

$$C \sim \left[\begin{array}{ccc|c} 1 & 5 & 7 & 15 \\ 0 & -7 & -10 & -19 \\ 0 & 1 & 2 & 5 \end{array}\right]$$

Applying, $R_1 \to R_1 - 5R_3$, $R_2 \to R_2 + 7R_3$

$$C \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -10 \\ 0 & 0 & 4 & 16 \\ 0 & 1 & 2 & 5 \end{array}\right]$$

Applying, $R_2 \to R_2 / 4$

$$C \sim \left[\begin{array}{ccc|c} 1 & 0 & -3 & -10 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 2 & 5 \end{array}\right]$$

Applying, $R_1 \to R_1 + 3R_3$, $R_3 \to R_3 - 2R_2$

$$C \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & -3 \end{array}\right]$$

Clearly $\rho(A) = 3$ and $\rho(C) = 3 \Rightarrow \rho(A) = \rho(C) = 3$ (No. of unknown variables)

$\therefore$ The system is consistent and having unique solutions.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ x \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$$

$\therefore$ $x = 2, y = -3, z = 4$

Hence the required solution is

$\boxed{x = 2, y = -3, z = 4}$ **Answer**

2. a) Find eigen values of the matrix $A=\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

Solution : The characteristic equation is

$$|A-\lambda I|=0$$

$$\Rightarrow \quad \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix}=0$$

$$\Rightarrow \quad (6-\lambda)\left[(3-\lambda)^2-1\right]+2\left[-2(3-\lambda)+2\right]+2\left[2-2(3-\lambda)\right]=0$$

$$\Rightarrow \quad (6-\lambda)\left[(3-\lambda-1)(3-\lambda+1)\right]+4\left[-3+\lambda+1\right]+4\left[1-3+\lambda\right]=0$$

$$\Rightarrow \quad (6-\lambda)\left[(2-\lambda)(4-\lambda)\right]+4\left[\lambda-2\right]+4\left[\lambda-2\right]=0$$

$$\Rightarrow \quad (\lambda-2)\left[-(6-\lambda)(4-\lambda)+4+4\right]=0$$

$$\Rightarrow \quad -(\lambda-2)\left[\lambda^2-10\lambda+16\right]=0$$

$$\Rightarrow \quad -(\lambda-2)(\lambda-2)(\lambda-8)=0$$

$$\Rightarrow \quad \boxed{\lambda=8,2,2}$$ **Answer**

b) Find inverse of the matrix using Cayley-Hamilton theorem

$$A=\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution : Given the matrix is

$$A=\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}$$

The characteristics equation is,

$$|A-\lambda I|=0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix}=0$$

$$\Rightarrow \quad (4-\lambda)\left[(1-\lambda)(1-\lambda)+4\right]-3\left[2(1-\lambda)+2\right]+1\left[4-1(1-\lambda)\right]=0$$

$\Rightarrow \quad (4-\lambda)[\lambda^2 - 2\lambda + 1 + 4] - 6[1 - \lambda + 1] + [4 - 1 + \lambda] = 0$

$\Rightarrow \quad (4-\lambda)[\lambda^2 - 2\lambda + 5] - 6[2 - \lambda] + [3 + \lambda] = 0$

$\Rightarrow \quad 4\lambda^2 - 8\lambda + 20 - \lambda^3 + 2\lambda^2 - 5\lambda - 12 + 6\lambda + 3 + \lambda = 0$

$\Rightarrow \quad -\lambda^3 + 6\lambda^2 - 6\lambda + 11 = 0$

$\Rightarrow \quad \lambda^3 - 6\lambda^2 + 6\lambda - 11 = 0 \qquad \ldots (1)$

This is required characteristic equation.

We know that by Cayley-Hamilton theorem every characteristic equation satisfy its characteristics equation, then from (1), we have

$$A^3 - 6A^2 + 6A - 11I = 0 \qquad \ldots(2)$$

Now, $A^2 = A.A = \begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix}$

Since $|A| \neq 0$, then A^{-1} is exist.

Equation (2) pre multiply by A^{-1}, we get

$$A^{-1}(A^3 - 6A^2 + 6A - 11I) = A^{-1}.0$$

$\Rightarrow \quad A^2 - 6A + 6I - 11A^{-1} = 0$

$\Rightarrow \quad = \begin{bmatrix} 23 & 17 & -1 \\ 8 & 3 & -2 \\ 9 & 7 & -2 \end{bmatrix} - 6\begin{bmatrix} 4 & 3 & 1 \\ 2 & 1 & -2 \\ 1 & 2 & 1 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\Rightarrow \quad = \begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}$

$\therefore \quad \boxed{A^{-1} = \frac{1}{11}\begin{bmatrix} 5 & -1 & -7 \\ -4 & 3 & 10 \\ 3 & -5 & -2 \end{bmatrix}}$ **Answer**

3. a) Solve the differential equation $x\frac{dy}{dx}+\cot y=0$

Solution : Given : $x\frac{dy}{dx}+\cot y=0$

$\Rightarrow \quad x\frac{dy}{dx}=-\cot y$

$\Rightarrow \quad x\frac{dy}{dx}=-\cot y$

$\Rightarrow \quad \frac{dy}{\cot y}=-\frac{dx}{x}$

$\Rightarrow \quad \tan y\,dy=-\frac{dy}{x}$

$\Rightarrow \quad \log\sec y=-\log x+\log c$

$\Rightarrow \quad \log\sec y=\log\left(\frac{c}{x}\right)$

$\Rightarrow \quad \sec y=\frac{c}{x}$

Thus, $\boxed{\sec y=\frac{c}{x}}$ **Answer**

b) Solve the differential equation $\frac{dy}{dx}+\frac{y}{x}=x^2$

Solution : Given LDE is,

$$\frac{dy}{dx}+\frac{y}{x}=x^2 \qquad \ldots(1)$$

Here, $P=1/x$ and $Q=x^2$

$\therefore \quad I.F.=e^{\int Pdx}=e^{\int(1/x)dx}e^{\int\log x}=x$

The solution is

$$y.(I.F.)=c+\int Q.(I.F.)dx$$

$\Rightarrow \quad y(x)=c+\int x^2.(x)dx$

$\Rightarrow \quad x.y=c+\frac{x^3}{3}$

Thus the required solution is

$\boxed{x.\,y=c+\frac{x^3}{3}}$ **Answer**

4. a) Solve the differential equation $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=5e^{3x}$

Solution : Given differential equation is

$$\frac{d^2y}{dx^2}+6\frac{dy}{dx}+9y=5e^{3x} \qquad \ldots (1)$$

$\Rightarrow \quad (D^2+6D+9)y=5e^{3x}$

The A.E. is

$$m^2+6m+9=0$$

$\Rightarrow \quad (m+3)^2=0$

$\Rightarrow \quad m=-3,\ -3$

$\therefore \quad C.F.=(C_1+xC_2)e^{-3x}$

Now, $\quad P.I.=\frac{1}{D^2+6D+9}5e^{3x}$

$\Rightarrow \quad =\frac{1}{(3)^2+6(3)+9}5e^{3x} \qquad$ Here $f(3)\neq 0$

$\Rightarrow \quad P.I.=\frac{5}{64}e^{3x}$

The required solution is,

$$y=C.F.+P.I.$$

$\Rightarrow \quad \boxed{y=(C_1+xC_2)e^{-3x}+\frac{5}{64}e^{3x}} \qquad$ **Answer**

b) Solve : $x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=\log x$

Solution : Given differential equation is

$$x^2\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=\log x \qquad \ldots (1)$$

This is homogeneous linear differential equation.

So put $\quad x=e^z$

$\Rightarrow \quad z=\log x$ and $\frac{dz}{dx}=\frac{1}{x}$

and $x\frac{d}{dx} \equiv D,\ x^2\frac{d^2}{dx^2} \equiv D(D-1)$ as $D \equiv \frac{d}{dz}$

then equation (1), becomes

$$[D(D-1)-D+1]y = z$$

$\Rightarrow$ $[D^2-2D+1]y = z$

The A.E. is,

$$m^2-2m+1=0$$

$\Rightarrow$ $(m-1)^2=0$

$\Rightarrow$ $m=1,1$

$\therefore$ $C.F. = (c_1+zc_2)\, e^z = [c_1+\log xc_2]x$

$$P.I. = \frac{1}{(D-1)^2}z$$

$$=[1-D]^{-2}z = [1+2D+3D^2+...]z$$

$\therefore$ The required solution is,

$$y = C.F. + P.I.$$

$\Rightarrow$ $\boxed{y = [c_1+\log xc_2]x + [\log x + 2]}$ **Answer**

5. a) Solve $\frac{dx}{dt}+y=\sin t$ and $\frac{dy}{dt}+x=\cos t$

Solution : Suppose $\frac{d}{dt} \equiv D$

Therefore, $Dx + y = \sin t$ … (1)

$x + Dy = \cos t$ … (2)

Eliminate y from equations (1) and (2), we get

$D^2x + Dy = D\sin t$ … (3)

$x + Dy = \cos t$ … (4)

Subtracting (4) from (3), we get

$$(D^2-1)x = D\sin t - \cos t$$

$\Rightarrow$ $(D^2-1)x = \cos t - \cos t = 0$

$\Rightarrow \quad \left(D^2-1\right)x=0 \qquad \ldots (5)$

The A.E. is, $\quad m^2-1=0$

$\Rightarrow \quad m=1,\ -1$

The Solution of equation (5) is

$$\boxed{x=c_1e^t+c_2e^{-t}}$$ **Answer**

Differentiating *w.r.t.*, "*t*" we get

$$\frac{dx}{dt}=c_1e^t-c_2e^{-t}$$

Given $\quad y=\dfrac{dx}{dt}+\sin t$

$\Rightarrow \quad =c_1e^t-c_2e^{-t}+\sin t$

$\Rightarrow \quad \boxed{y=c_1e^t-c_2e^{-t}+\sin t}$ **Answer**

b) Solve the differential equation $\dfrac{d^2y}{dx^2}-2\tan x\dfrac{dy}{dx}-5y=0$ **by reducing it in normal form.**

Solution : Given the differential equation is,

$$\frac{d^2y}{dx^2}-2\tan x\frac{dy}{dx}-5y=0 \qquad \ldots.(1)$$

Here, $P=-2\tan x, Q=-5$ and $R=0$

Now this problem solve by Removable of first derivative method.

Suppose the complete solution is,

$$y=v\,y_1 \qquad \ldots.(2)$$

Where v is a function of x only.

Now we can find the value of y_1 such as

$$y_1=e^{-\frac{1}{2}\int Pdx}=e^{-\frac{1}{2}\int(-2\tan x)dx}=e^{\log\sec x}=\sec x$$

and $\quad Q_1=Q-\dfrac{1}{4}P^2-\dfrac{1}{2}\dfrac{dP}{dx}=-5-\dfrac{1}{4}\left(-2\tan x\right)^2-\dfrac{1}{2}\left(-2\sec^2 x\right)=-5-\tan^2 x+\sec^2 x=-4$

and $\quad R_1=\dfrac{R}{y_1}=0$

The normal form of equation is,

$$\frac{d^2v}{dx^2}+Q_1v=0$$

$\Rightarrow \quad \frac{d^2v}{dx^2}-4v=0 \qquad \text{…. (3)}$

This is LDR of higher order.

The A.E. is

$$m^2-4=0$$

$\Rightarrow \quad m=2,-2$

$\therefore \quad v=c_1e^{2x}+c_2e^{-2x}$

Putting in equation (2), which our complete solution

$$\boxed{y=\left[c_1e^{2x}+c_2e^{-2x}\right]\sec x}$$ **Answer**

6. a) Solve the differential equation $\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=\frac{e^{3x}}{x^2}$ using variation of parameter.

Solution : Given differential equation is

$\frac{d^2y}{dx^2}-6\frac{dy}{dx}+9y=\frac{e^{3x}}{x^2} \qquad \text{….(1)}$

$\Rightarrow \quad \left(D^2-6D+9\right)y=\frac{e^{3x}}{x^2}$

The A.E. is

$$m^2-6m+9=0$$

$\Rightarrow \quad (m-3)^2=0$

$\Rightarrow \quad m=3\ 3$

$\therefore \quad y_c=c_1e^{3x}+C_2\left(xe^{3x}\right)$

Say, $u=e^{3x}$ and $v=xe^{3x}$

$\Rightarrow \quad u'=3e^{3x}$ and $v'=e^{3x}(3x+1)$

Now, $w=\begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x}(3x+1) \end{vmatrix}=e^{6x}(3x+1-3x)=e^{6x}\neq 0$

Suppose the complete solution of equation (1) is

$y=Au+By=A\left(e^{3x}\right)+B\left(x.e^{3x}\right) \qquad \text{…(2)}$

Where A and B determine by the formula,

$$A = \int \left(-\frac{v.R}{w} \right) dx + c_1 = -\int \left(\frac{xe^{3x}}{e^{6x}} \times \frac{e^{3x}}{x^2} \right) dx + c_1 \qquad \left[\Theta \; R = \frac{e^{3x}}{x^2} \right]$$

$$\Rightarrow \qquad \boxed{A = -\int \frac{1}{x} dx + c_1 = -\log x + c_1}$$

and $$B = \int \left(\frac{u.R}{w} \right) dx + c_2 = \int \left(\frac{e^{3x}}{e^{6x}} \times \frac{e^{3x}}{x^2} \right) dx + c_2$$

$$\Rightarrow \qquad B = \int \frac{1}{x^2} dx + c_2 = -\frac{1}{x} + c_2$$

$$\Rightarrow \qquad \boxed{B = -\frac{1}{x} + c_2}$$

Putting the values of A and B in equation (2), we get

$$y = [-\log x + c_1] e^{3x} + \left[-\frac{1}{x} + c_2 \right] x.e^{3x}$$

$$\Rightarrow \qquad \boxed{y = C_1 e^{3x} + C_2 \left(xe^{3x} \right) - e^{3x} (\log x + 1)}$$ **Answer**

b) Form the partial differential equation from the relation $z = (x+a)(y+b)$, **a and b are constant.**

Solution : Given relation is, $z = (x+a)(y+b)$ …(1)

Differentiating (1) w.r.t. *x* and *y* partially, we get

$$\frac{\partial z}{\partial x} = y + b = p \qquad \text{…(2)}$$

and $$\frac{\partial z}{\partial y} = x + a = q \qquad \text{…(3)}$$

Now equation (2) and (3) are multiplied together, we get

$$p.q = (x+a)(y+b)$$

$$= \qquad p.q = z$$

Thus required PDE is

$$\boxed{p.q. = z}$$ **Answer**

7. a) Solve the partial differential equation $yq - xp = z$

Solution : Given PDE is

$$yq - xp = z \qquad \ldots(1)$$

Here, $P = -x$, $Q = y$ and $R = z$

The Lagarnge's A.E. is

$$\frac{dx}{-x} = \frac{dy}{y} = \frac{dz}{z}$$

Taking first two ratio's

$$\frac{dx}{-x} = \frac{dy}{y}$$

Integrating on both sides, we get

$$-\log x = \log y + \log c_1$$

$$\log\left(\frac{1}{x}\right) = \log\left(y\,c_1\right)$$

$$\frac{1}{x} = y\,c_1$$

$$\frac{1}{x\,y} = c_1$$

Taking Last two ratio's

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating on both sides, we get

$$\log y = \log z + \log c_2$$

$$\log y = \log\left(zc_2\right)$$

$$y = z\,c_2$$

$$\frac{y}{z} = c_2$$

The general solution is

$$\boxed{\phi\left(\frac{1}{xy}, \frac{y}{z}\right) = 0}$$ **Answer**

b) Solve the partial differential equation

$$\frac{\partial^2 z}{\partial x^2}+4\frac{\partial^2 z}{\partial x \partial y}-5\frac{\partial^2 z}{\partial y^2}=0$$

Solution : Give PDE is,

$$\frac{\partial^2 z}{\partial x^2}+4\frac{\partial^2 z}{\partial x \partial y}-5\frac{\partial^2 z}{\partial y^2}=0 \qquad \text{....(1)}$$

$$= \qquad \left(D^2+4DD'-5D'^2\right)z=0$$

The A.E. is,

$$m^2+4m-5=0$$

$$(m+5)(m-1)=0$$

$$m=-5,\ 1$$

The complete solution is,

$$\boxed{z=\phi_1(y-5x)+\phi_2(y+x)} \qquad \textbf{Answer}$$

8. a) Solve the partial differential equation

$$\frac{\partial^2 z}{\partial x^2}+3\frac{\partial^2 z}{\partial x \partial y}+2\frac{\partial^2 z}{\partial y^2}=x+y$$

Solution : Given PDE is,

$$\frac{\partial^2 z}{\partial x^2}+3\frac{\partial^2 z}{\partial x \partial y}+2\frac{\partial^2 z}{\partial y^2}=x+y \qquad \text{...(1)}$$

$$\Rightarrow \qquad \left(D^2+3DD'+2D'2\right)z=x+y$$

The A.E. is,

$$m^2+2m+2=0$$

$$(m+2)(m+1)=0$$

$$m=-2,-1$$

$$\therefore \qquad C.F.=\phi_1(y-2x)+\phi_2(y-x)$$

Now, $P.I.=\dfrac{1}{D^2+3DD'+2D'^2}(x+y)$

$$=\frac{1}{(1)^2+3(1)(1)+2(1)^2}\int\int v(dv)^2 \text{ where } v=x+y \text{ and } f(1,1)\neq 0 \text{ [By Short-cut Method]}$$

$$= \frac{1}{6}\left(\frac{v^3}{6}\right)$$

$$P.I. = \frac{(x+y)^3}{36}$$

The complete solution is,

$$z = C.F. + P.I.$$

$$\therefore \quad \boxed{z = \phi_1(y-2x) + \phi_2(y-x) + \frac{(x+y)^3}{6}}$$

b) Solve the equation $zp + yq = x$

Solution : Given PDE is

$$zp + yq = x \qquad \ldots(1)$$

Here, P=z, Q=y and R = x

The Lagarnge's A.E. is

$$\frac{dx}{z} = \frac{dy}{y} = \frac{dz}{x}$$

Taking first and Last ratio's

$$\frac{dx}{z} = \frac{dz}{x}$$

$$x\,dx = z\,dz$$

Integrating on both sides, we get

$$\frac{x^2}{2} = \frac{z^2}{2} + \frac{c_1^{\,2}}{2}$$

$$x^2 - z^2 = c_1^{\,2} \qquad \ldots.(2)$$

Taking Last two ratio's

$$\frac{dy}{y} = \frac{dz}{x}$$

$$\Rightarrow \quad \frac{dy}{y} = \frac{dz}{\sqrt{z^2 + c_1^2}} \qquad \text{From (2)} \qquad \left[\Theta\ x^2 - z^2 = c_1^{\,2}\ i.e. x^2 = z^2 + c_1^{\,2}\right]$$

Integrating on both sides, we get

$$\log y = \log\left[z + \sqrt{z^2 + c_1^{\,2}}\ \right] + \log c_2$$

$$y = c_2\left[z + \sqrt{z^2 + c_1^{\ 2}}\right]$$

$$\frac{y}{z + \sqrt{z^2 + c_1^{\ 2}}} = c_2$$

$$\frac{y}{z + x} = c_2$$

The general solution is

$$\boxed{\phi\left(x^2 - z^2, \frac{y}{z + x}\right) = 0}$$ **Answer**