1. a) Discuss the maxima and minima of the function $u = x^3y^2(1-x-y)$

Solution: Given the equation of curve is

$$u = x^3y^2(1-x-y) \qquad \dots (1)$$

Partially differentiate w.r.t., x and y we get

$$\frac{\partial u}{\partial x} = x^3y^2(-1) + 3x^2y^2(1-x-y) = -x^3y^2 + 3x^2y^2(1-x-y) \qquad \dots (2)$$

and $$\frac{\partial u}{\partial y} = x^3y^2(-1) + 2x^3y(1-x-y) = -x^3y^2 + 2x^3y(1-x-y) \qquad \dots (3)$$

Taking, $\frac{\partial u}{\partial x} = 0$ http://www.rgpvonline.com

$$\Rightarrow \quad -x^3y^2 + 3x^2y^2(1-x-y) = 0$$

$$\Rightarrow \quad 3x^2y^2(1-x-y) = x^3y^2 \qquad \dots (4)$$

and $\frac{\partial u}{\partial y} = 0$

$$\Rightarrow \quad -x^3y^2 + 2x^3y(1-x-y)$$

$$\Rightarrow \quad 2x^3y(1-x-y) = x^3y^2 \qquad \dots (5)$$

From (4) and (5), we get

$$2x^3y(1-x-y) = 3x^2y^2(1-x-y)$$

$$\Rightarrow \quad 2x = 3y$$

$$\Rightarrow \quad y = \frac{2}{3}x \qquad \dots (6)$$

From equation (4), we get

$$3x^2y^2(1-x-y) = x^3y^2$$

$$\Rightarrow \quad 3(1-x-y) = x$$

$$\Rightarrow \quad 3\left(1-x-\frac{2}{3}x\right) = x$$

$$\Rightarrow \quad 3-5x = x$$

$$\Rightarrow \quad x = \frac{1}{2}$$

Putting in equation (6), we get

$$y = \frac{2}{3}\left(\frac{1}{2}\right) = \frac{1}{3}$$

The required stationary point is $\left(\frac{1}{2}, \frac{1}{3}\right)$

From equation (2), we have

$$\frac{\partial u}{\partial x}=-x^3y^2+3x^2y^2-3x^3y^2-3x^2y^3=3x^2y^2-4x^3y^2-3x^2y^3$$

Partially differentiate w.r.t., x and y we get

$$r=\frac{\partial^2 u}{\partial x^2}=6xy^2-12x^2y^2-6xy^3$$

and $$s=\frac{\partial^2 u}{\partial y\,\partial x}=6x^2y-8x^3y-9x^2y^2$$

From equation (3), we have

$$\frac{\partial u}{\partial y}=-x^3y^2+2x^3y-2x^4y-2x^3y^2=2x^3y-2x^4y-3x^3y^2$$

Partially differentiate w.r.t., y we get

$$t=\frac{\partial^2 u}{\partial y^2}=2x^3-2x^4-6x^3y$$

Putting $x=\frac{1}{2}$ and $y=\frac{1}{3}$

$$r=6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^2-12\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2-6\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)^3=-\frac{1}{9}>0$$

and $$s=6\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)-8\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)-9\left(\frac{1}{2}\right)^2\left(\frac{1}{3}\right)^2=-\frac{1}{12}$$

and $$t=2\left(\frac{1}{2}\right)^3-2\left(\frac{1}{2}\right)^4-6\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)=-\frac{1}{8}$$

Now, $rt-s^2=\left(-\frac{1}{9}\right)\left(-\frac{1}{8}\right)-\left(-\frac{1}{12}\right)^2=\frac{1}{144}>0$

Since, $r>0$ and $rt-s^2>0$, therefore given function have minimum value.

Now, $u_{\min}\left(\frac{1}{2},\frac{1}{3}\right)=\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(1-\frac{1}{2}-\frac{1}{3}\right)=\frac{1}{432}$ **Answer**

b) Expand $\log_e x$ in power of x and hence evaluate $\log_e(1.1)$ correct to four decimal places.

Solution: Suppose $f(x) = \log x$

Successive differentiation w.r.t., x, we get

$$f'(x) = \frac{1}{x},\ f''(x) = -\frac{1}{x^2},\quad f'''(x) = \frac{2}{x^3} \text{ and } f^{iv}(x) = -\frac{6}{x^4}$$

Putting x = 1, we get

$$f(1) = \log 1 = 0,\ f'(1) = 1,\ f''(1) = -1,\quad f'''(1) = 2 \text{ and } f^{iv}(1) = -6$$

We know that Taylor series in power of (x−1), we get

$$f(x) = f(1) + \frac{x-1}{\lfloor 1}f'(1) + \frac{(x-1)^2}{\lfloor 2}f''(1) + \frac{(x-1)^3}{\lfloor 3}f'''(1) + \frac{(x-1)^4}{\lfloor 4}f^{iv}(1) + \ldots$$

$$\Rightarrow \quad \log x = 0 + \frac{x-1}{\lfloor 1}(1) + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{6}(2) + \frac{(x-1)^4}{24}(-6) + \ldots$$

$$\Rightarrow \quad \log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

Putting $x = 1.1$, we get

$$\log(1.1) = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \ldots$$

$$\Rightarrow \quad = 0.1 - 0.005 + 0.00033 - 0.000025$$

$$\Rightarrow \quad \boxed{\log(1.1) = 0.095305}$$

Answer

2. a) Verify Lagrange's Mean value theorem for the function $f(x)=2x^2-7x+10$ in [2, 5]

Solution: Given the function is

$$f(x)=2x^2-7x+10 \qquad \dots (1)$$

(i). Putting $x=a=2$ and $x=b=5$, we get

$$f(2)=2(2)^2-7(2)+10=4$$

and $f(5)=2(5)^2-7(5)+10=25$

Clearly $f(2)\neq f(5)$

(ii). Since $f(x)$ is polynomial function in x, then $f(x)$ is continuous in [2, 5].

(iii). Since $f(x)$ is polynomial function in x, then it can be differentiate such that

$$f'(x)=4x-7$$

then by LMVT $\exists$ at least $c\in(2,5)$ such that

$$f'(c)=\frac{f(5)-f(2)}{5-2}$$

$\Rightarrow$ $$4c-7=\frac{25-4}{5-2}$$

$\Rightarrow$ $$4c=14 \Rightarrow c=3.75\in(2,5)$$

Hence Lagrange's mean value theorem is verified for $f(x)$ in [2, 5].

b) If $u = f(y-z, z-x, x-y)$, then prove that $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$

Solution: Suppose $X = y-z,\quad Y = z-x, Z = x-y$, then $u = f(X,Y,Z)$

Therefore u is composite function x, y and z respectively.

We have $X = y-z,\quad Y = z-x, Z = x-y$

Partially differentiate $w.r.t.$ x, y and z respectively, we get

$$\frac{\partial X}{\partial x}=0, \frac{\partial X}{\partial y}=1, \frac{\partial X}{\partial z}=-1$$ http://www.rgpvonline.com

And $$\frac{\partial Y}{\partial x}=-1, \frac{\partial Y}{\partial y}=0, \frac{\partial Y}{\partial z}=1,\ \frac{\partial Z}{\partial x}=1, \frac{\partial Z}{\partial y}=-1, \frac{\partial Z}{\partial z}=0$$

Now, $$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial X}.\frac{\partial X}{\partial x}+\frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial x}+\frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial x}$$

$$=\frac{\partial u}{\partial X}.(0)+\frac{\partial u}{\partial Y}.(-1)+\frac{\partial u}{\partial Z}.(1)=-\frac{\partial u}{\partial Y}.+\frac{\partial u}{\partial Z} \quad \dots (1)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial X}.\frac{\partial X}{\partial y}+\frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial y}+\frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial y}$$

$$=\frac{\partial u}{\partial X}.(1)+\frac{\partial u}{\partial Y}.(0)+\frac{\partial u}{\partial Z}.(-1)=\frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z} \quad \dots (2)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial X}.\frac{\partial X}{\partial z}+\frac{\partial u}{\partial Y}.\frac{\partial Y}{\partial z}+\frac{\partial u}{\partial Z}.\frac{\partial Z}{\partial z}$$

$$=\frac{\partial u}{\partial X}.(-1)+\frac{\partial u}{\partial Y}.(1)+\frac{\partial u}{\partial Z}.(0)=-\frac{\partial u}{\partial X}+\frac{\partial u}{\partial Y} \quad \dots (3)$$

Adding (1), (2) and (3), we get

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=-\frac{\partial u}{\partial Y}.+\frac{\partial u}{\partial Z}+\frac{\partial u}{\partial X}-\frac{\partial u}{\partial Z}-\frac{\partial u}{\partial X}+\frac{\partial u}{\partial Y}=0$$

$\Rightarrow$ $$\boxed{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0}$$ **Hence proved**

3. a) Evaluate $\int_a^b x^2\,dx$ on limit of sum.

Solution: Given $f(x)=x^2$ and $nh=b-a$

We know that by definition of definite integral as limit of sum,

$$\int_a^b f(x)\,dx=\lim_{h\to 0} h\left[f(a)+f(a+h)+f(a+2h)+....+f(a+(n-1)h\right]$$

$$\Rightarrow \quad \int_a^b x^2\,dx=\lim_{h\to 0} h\left[a^2+(a+h)^2+(a+2h)^2+....+\{a+(n-1)h\}^2\right]$$

$$\Rightarrow \quad =\lim_{h\to 0} h\left[a^2+(a^2+h^2+2ah)+(a^2+2^2h^2+4ah)+....+\left\{a^2+(n-1)^2h^2+2a(n-1)h\right\}\right]$$

$$\Rightarrow \quad =\lim_{h\to 0} h\left[a^2(1+1+1+..n\,times)+h^2\left(1^2+2^2+...+(n-1)^2\right)+2ah\left(1+2+...+(n-1)\right)\right]$$

http://www.rgpvonline.com $\left[\because\ 1+2+3+...+n\,times=\sum n=\frac{n(n+1)}{2}\right]$

$$\Rightarrow \quad =\lim_{h\to 0} h\left[na^2+h^2\frac{(n-1)n(2n-1)}{6}+2ah\frac{(n-1)n}{2}\right]$$

$$\Rightarrow \quad =\lim_{h\to 0}\left[nha^2+\frac{(nh-h)nh(2nh-h)}{6}+2a\frac{(nh-h)nh}{2}\right]$$

$$\Rightarrow \quad =\lim_{h\to 0}\left[(b-a)a^2+\frac{(b-a-h)(b-a)(2(b-a)-h)}{6}+a(b-a-h)(b-a)\right]$$

$$\Rightarrow \quad =(b-a)\lim_{h\to 0}\left[a^2+\frac{(b-a-h)(2(b-a)-h)}{6}+a(b-a-h)\right]$$

$$\Rightarrow \quad =(b-a)\left[a^2+\frac{(b-a-0)(2(b-a)-0)}{6}+a(b-a-0)\right]=(b-a)\left[a^2+\frac{(b-a)^2}{3}+a(b-a)\right]$$

$$\Rightarrow \quad =\frac{(b-a)}{3}\left[3a^2+b^2+a^2-2ba+3ab-3a^2\right]=\frac{(b-a)}{3}\left[b^2+a^2+ab\right]$$

$$\Rightarrow \quad =\frac{b^3-a^3}{3}$$

Thus $\boxed{\int_a^b x^2\,dx=\frac{b^3-a^3}{3}}$ **Answer**

b) Prove that $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$

Solution: Suppose $I = \int_0^\infty e^{-x^2}\,dx$

Putting, $x^2 = t \Rightarrow x = \sqrt{t}$

$\therefore$ $dx = \frac{1}{2\sqrt{t}}\,dt$

Now, $I = \int_0^\infty e^{-t}\,\frac{1}{2\sqrt{t}}\,dt$

$$= \frac{1}{2}\int_0^\infty e^{-t}\,t^{-1/2}dt = \frac{1}{2}\int_0^\infty e^{-t}\,t^{(1/2)-1}dt$$

$$= \frac{1}{2}\left\lceil\frac{1}{2}\right. = \frac{\sqrt{\pi}}{2}$$

Hence Proved

4. a) Evaluate: $\int_0^2 \int_0^1 (x^2 + y^2)\, dx\, dy$

Solution:

$$\int_0^2 \int_0^1 (x^2 + y^2)\, dx\, dy = \int_0^2 \left[x^2 y + \left(\frac{y^3}{3} \right) \right]_0^1 dx$$

$$= \int_0^2 \left[\left(x^2 + \frac{1}{3} \right) - (0+0) \right] dx = \int_0^2 \left(x^2 + \frac{1}{3} \right) dx$$

$$= \left[\frac{x^3}{3} + \frac{x}{3} \right]_0^2 = \left(\frac{8}{3} + \frac{2}{3} \right) - (0+0) = \frac{10}{3}$$ **Answer**

b) Evaluate $\int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

Solution: Suppose $I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

$$\Rightarrow \quad = \int_0^a \int_0^x e^{x+y}\left[\int_0^{x+y} e^{z}\, dz\right] dx\, dy$$

$$\Rightarrow \quad = \int_0^a \int_0^x e^{x+y}\left[e^{z}\right]_0^{x+y} dx\, dy$$

$$\Rightarrow \quad = \int_0^a \int_0^x e^{x+y}\left[e^{x+y} - 1\right] dx\, dy$$

$$\Rightarrow \quad = \int_0^a e^{x}\left[\int_0^x \left(e^{x+2y} - e^{y}\right) dy\right] dx$$

$$\Rightarrow \quad = \int_0^a e^{x}\left[\frac{e^{x+2y}}{2} - e^{y}\right]_0^x dx$$

$$\Rightarrow \quad = \int_0^a e^{x}\left[\left\{\frac{e^{3x}}{2} - e^{x}\right\} - \left\{\frac{e^{x}}{2} - 1\right\}\right] dx$$

$$\Rightarrow \quad = \int_0^a e^{x}\left[\frac{e^{3x}}{2} - \frac{3}{2}e^{x} + 1\right] dx = \int_0^a \left[\frac{e^{4x}}{2} - \frac{3}{2}e^{2x} + e^{x}\right] dx$$

$$\Rightarrow \quad = \left[\frac{e^{4x}}{8} - \frac{3}{4}e^{2x} + e^{x}\right]_0^a = \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^{a} - \left(\frac{1}{8} - \frac{3}{4} + 1\right)$$

$$\Rightarrow \quad = \frac{e^{4a}}{8} - \frac{3}{4}e^{2a} + e^{a} - \frac{3}{8}$$

Answer

5. a) Test the convergence of the following series.

$$\sum u_n = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

Solution: The given series can be written as

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} + \dots$$

This is geometric series and common ratio is $r = \frac{1}{2} < 1$

therefore the series is convergent.

b) Express $f(x) = x$ as a half range cosine series in the interval $0 < x < 2$

Solution: Given: $f(x) = x, \quad 0 < x < 2$ ... (1)

Here, $L = 2$

Suppose the Half range cosine series of $f(x)$ is,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

$$\Rightarrow \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} b_n \cos\left(\frac{n\pi x}{2}\right) \qquad [\text{Since } L = \pi] \qquad \text{... (2)}$$

Now, $a_0 = \frac{2}{L}\int_0^L f(x)\,dx$

$$\Rightarrow \quad = \frac{2}{2}\int_0^2 x\,dx$$

$$\Rightarrow \quad = 1\left[\frac{x^2}{2}\right]_0^2 = \frac{1}{2}\left[2^2 - 0\right] = 2$$

$$\Rightarrow \quad \boxed{a_0 = 2}$$

and $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$$\Rightarrow \quad a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$\Rightarrow \quad = \left[x\left(\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\right) - 1\left(-\frac{4}{n^2\pi^2}\right)\cos\left(\frac{n\pi x}{2}\right)\right]_0^2$$

$$\Rightarrow \quad = \left[\left\{0 + \frac{4}{n^2\pi^2}(-1)^n\right\} - \left\{0 + \frac{4}{n^2\pi^2}\right\}\right]$$

$$\Rightarrow \quad = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]$$

Putting in equation (1), we get

$$\boxed{f(x) = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[(-1)^n - 1\right]}{n^2}\cos\left(\frac{n\pi x}{2}\right)}$$

Answer

6. a) Show that the mapping $T: R^2 \to R^3$ defined by

$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_2)$ is a linear transformation.

Solution: Given the mapping is, $T: R^2 \to R^3$ such that

$$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_2) \qquad \ldots (1)$$

Let $\alpha = (a_1, b_1) \Rightarrow T(\alpha) = T(a_1, b_1) = (a_1 + b_1, a_1 - b_1, b_1) \qquad \ldots (2)$

and $\beta = (a_2, b_2) \Rightarrow T(\beta) = T(a_2, b_2) = (a_2 + b_2, a_2 - b_2, b_2) \qquad \ldots (3)$

Let $a, b \in R$ and $\alpha, \beta \in R^2$, then

$$T(a\alpha + b\beta) = T[a(a_1, b_1) + b(a_2, b_2)]$$

$$= T(aa_1 + ba_2, ab_1 + bb_2)$$

$$= [(aa_1 + ba_2) + (ab_1 + bb_2), (aa_1 + ba_2) - (ab_1 + bb_2), ab_1 + bb_2] \qquad \text{[From (1)]}$$

$$= [a(a_1 + b_1) + b(a_2 + b_2), a(a_1 - b_1) + b(a_2 - b_2), ab_1 + bb_2]$$

$$= a(a_1 + b_1, a_1 - b_1, b_1) + b(a_2 + b_2, a_2 - b_2, b_2)$$

$$= aT(\alpha) + T(\beta) \qquad \text{[From (2) and (3)]}$$

$\therefore$ ***T* is a linear transformation.** **Proved**

b) Show that the set S of vectors (1, 0, 0), (1, 1, 0) and (1, 1, 1) is linearly independent.

Solution: Given: $\alpha_1 = (1, 0, 0)$, $\alpha_2 = (1, 1, 0)$ and $\alpha_3 = (1, 1, 1)$ and let $a_1, a_2, a_3 \in R$, such that

$$a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = \vec{0}$$

$$\Rightarrow \quad a_1(1, 0, 0) + a_2(1, 1, 0) + a_3(1, 1, 1) = \vec{0}$$

$$\Rightarrow \quad (a_1, 0, 0) + (a_2, a_2, 0) + (a_3, a_3, a_3) = \vec{0}$$

$$\Rightarrow \quad (a_1 + a_2 + a_3, a_2 + a_3, a_3) = (0, 0, 0)$$

Equating both sides, we get

$$a_1 + a_2 + a_3 = 0$$

$$a_2 + a_3 = 0$$

and

$$a_3 = 0$$

$\Rightarrow \quad a_1 = 0, a_2 = 0$ and $a_3 = 0$

Hence, the given set of vectors is **linearly independent set**.

7. a) Find rank if the matrix $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Solution: Now will find the rank of matrix by Echelon form

Given $A = \begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 1 & 3 & 4 & 5 \end{bmatrix}$

Applying, $R_2 \to R_2 - 2R_1, R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

Applying, $R_3 \to R_3 + R_2$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly number of non-zero two rows, then

$\rho\,(A) = 2$ **Answer**

b) Solve the system of equations $3x+3y+2z=1$, $x+2y=4$, $10y+3z=-2$ and $2x-3y-z=5$

Solution: Given the system of equation is

$$\begin{bmatrix} 3 & 3 & 2 \\ 1 & 2 & 0 \\ 0 & 10 & 3 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ -2 \\ 5 \end{bmatrix}$$

$\Rightarrow$ $$AX = B$$

This is Non-Homogeneous system of equation, and then augmented matrix is

$$C = [A \mid B] = \left[\begin{array}{ccc|c} 3 & 3 & 2 & 1 \\ 1 & 2 & 0 & 4 \\ 0 & 10 & 3 & -2 \\ 2 & -3 & -1 & 5 \end{array}\right]$$

Applying, $R_1 \leftrightarrow R_2$

$$C \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 3 & 3 & 2 & 1 \\ 5 & 3 & 14 & 4 \\ 2 & 1 & 6 & 2 \end{array}\right]$$

Applying, $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 5R_1, R_4 \to R_4 - 2R_1$

$$C \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 10 & 3 & -2 \\ 0 & -7 & -1 & -3 \end{array}\right]$$

Applying, $R_3 \to R_3 + R_2$

$$C \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 7 & 5 & -13 \\ 0 & -7 & -1 & -3 \end{array}\right]$$

Applying, $R_4 \to R_4 + R_3$

$$C \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 7 & 5 & -13 \\ 0 & 0 & 4 & -16 \end{array}\right]$$

Applying, $R_3 \to R_3 + \frac{7}{3}R_2,\ R_4 \to \frac{R_4}{4}$

$$C \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 29/3 & -116/3 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

Applying, $R_3 \to \frac{3}{29} R_3$

$$C \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & -4 \end{array}\right]$$

Applying, $R_4 \to R_4 - R_3$

$$C \sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & -3 & 2 & -11 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Clearly, $\rho(A) = \rho(C) = 3$ (No of variables)

Hence, the system is consistent and having unique solution.

$$\therefore \quad \begin{bmatrix} 1 & 2 & 0 \\ 0 & -3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -11 \\ -4 \end{bmatrix}$$

$\Rightarrow \quad x + 2y = 4 \quad \ldots (1)$

$-3y + 2z = -11 \quad \ldots (2)$

$z = -4 \quad \ldots (3)$

Putting the value of z in equation (2), we get

$$y = 1$$

Putting the value of y and z in equation (1), we get

$$x = 2$$

Thus, $x = 2, y = 1$ *and* $z = -4$ **Answer**

8. a) Find the Eigen values of the matrix

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Solution: The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (6-\lambda)\left[(3-\lambda)^2 - 1\right] + 2\left[-2(3-\lambda)+2\right] + 2\left[2-2(3-\lambda)\right] = 0$$

$$\Rightarrow \quad (6-\lambda)\left[(3-\lambda-1)(3-\lambda+1)\right] + 4\left[-3+\lambda+1\right] + 4\left[1-3+\lambda\right] = 0$$

$$\Rightarrow \quad (6-\lambda)\left[(2-\lambda)(4-\lambda)\right] + 4\left[\lambda-2\right] + 4\left[\lambda-2\right] = 0$$

$$\Rightarrow \quad (\lambda-2)\left[-(6-\lambda)(4-\lambda)+4+4\right] = 0$$

$$\Rightarrow \quad -(\lambda-2)\left[\lambda^2 - 10\lambda + 16\right] = 0$$

$$\Rightarrow \quad -(\lambda-2)(\lambda-2)(\lambda-8) = 0$$

$$\Rightarrow \quad \lambda = 8, 2, 2$$ **Answer**

b) Verify Cayley-Hamilton's theorem for the matrix $\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

Solution: **Solution**: Given the matrix is

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

The characteristics equation is,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $$(1-\lambda)\left[-(1-\lambda^2)-3\right]-2[-1-\lambda-1]-2[3-1+\lambda]=0$$

$\Rightarrow$ $$(1-\lambda)\left[\lambda^2-4\right]-2[-2-\lambda]-2[2+\lambda]=0$$

$\Rightarrow$ $$\lambda^2-4-\lambda^3+4\lambda+2[2+\lambda]-2[2+\lambda]=0$$

$\Rightarrow$ $$\lambda^3-\lambda^2-4\lambda+4=0 \qquad \ldots(1)$$

This is required characteristic equation. http://www.rgpvonline.com

Verification of Cayley-Hamilton theorem

By Cayley-Hamilton theorem every characteristic equation satisfy its characteristics equation, then from (1), we Have

$$A^3 - A^2 - 4A + 4I = 0 \qquad \ldots (2)$$

Now, $$A^2 = A.A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} . \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix}$$

and $$A^3 = A^2.A = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix}$$

$$L.H.S. = A^3 - A^2 - 4A + 4I$$

$$= \begin{bmatrix} 1 & 6 & -6 \\ 7 & 6 & 2 \\ 7 & 14 & -6 \end{bmatrix} - \begin{bmatrix} 1 & -2 & 2 \\ 3 & 6 & -2 \\ 3 & 2 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} + 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R.H.S.$$

Hence verify Cayley-Hamilton theorem.