RGPV SOLUTION MA-111 ENGINEERING MATHEMATICS 2 DEC-2017

1. (a) Find rank and nullity of the following matrix by reducing it to the normal form

$$\begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$$

Solution: Given the matrix is

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 4 & 10 \\ 3 & 8 & 4 \end{bmatrix}$$

Applying, $R_2 \to R_2 - 2R_1, R_3 \to R_3 - 3R_1$

$$A \sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & -2 & 4 \\ 0 & -1 & -5 \end{bmatrix}$$

Applying, $C_2 \to C_2 - 3C_1, C_3 \to RC_3 - 3C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 4 \\ 0 & -1 & -5 \end{bmatrix}$$

Applying, $R_2 \leftrightarrow R_3$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & -2 & 4 \end{bmatrix}$$

Applying, $R_3 \to R_3 + 2R_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 14 \end{bmatrix}$$

Applying, $C_3 \to C_3 - 5C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 14 \end{bmatrix}$$

Applying, $C_3 \to C_3 / 14$

$$A \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 14 \end{bmatrix} = [I_3]$$

Clearly this is required normal form and rank $\rho(A) = 3$ **Answer**

Nullity N(A) = order of square matrix-Rank of Matrix

N(A) = 3 – 3 = 0 **Answer**

(b) Examine whether the following equations are consistent and solve them if they are consistent.

$$2x - y + 3z = 0$$

$$3x + 2y + z = 0$$

$$x - 4y + 5z = 0$$

Solution : Given the system of equation is

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ $$AX = B$$

This is Homogeneous system of equation, and then augmented matrix is

$$C = [A:B] = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 0 \\ 3 & 2 & 1 & 0 \\ 1 & -4 & 5 & 0 \end{array}\right]$$

Applying, $R_1 \leftrightarrow R_3$

$$C \sim \left[\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 3 & 2 & 1 & 0 \\ 2 & -1 & 3 & 0 \end{array}\right]$$

Applying, $R_2 \to R_2 - 3R_1, R_3 \to R_3 - 2R_1$

$$C \sim \left[\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 0 & 14 & -14 & 0 \\ 0 & 7 & -7 & 0 \end{array}\right]$$

Applying, $R_2 \to R_2 / 14$

$$C \sim \left[\begin{array}{ccc|c} 1 & -4 & 5 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 7 & -7 & 0 \end{array}\right]$$

Applying, $R_1 \to R_1 + 4R_2, R_3 \to R_3 - 7R_2$

$$C \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Clearly $\rho(A) = 2$ and $\rho(C) = 2 \Rightarrow \rho(A) = \rho(C) = 2 < 3$ (No. of unknown variables)

$\therefore$ The system is consistent and having infinite many solutions.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + z = 0$$

$$y - z = 0$$

Taking, z = k, then we get x = k and y = k

Hence the required solution is

$\boxed{x = -k, y = k \text{ and } z = k}$ **Answer**

2. (a) Find Eigen values and Eigen vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution : The characteristic equation is

$$|A - \lambda l| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad (8-\lambda)[7-\lambda(3-\lambda)-16] + 6[-6(3-\lambda)+8] + 2[24-2(7-\lambda] = 0$$

$$\Rightarrow \quad (8-\lambda)[21-10\lambda+\lambda^2-16] + 6[-18+6\lambda+8] + 2[24-14+2\lambda] = 0$$

$$\Rightarrow \quad (8-\lambda)[\lambda^2-10\lambda+5] + 6[-10+6\lambda] + 2[10+2\lambda] = 0$$

$$\Rightarrow \quad 8\lambda^2 - 80\lambda + 40 - \lambda^3 + 10\lambda^2 - 5\lambda - 60 + 36\lambda + 20 + 4\lambda = 0$$

$$\Rightarrow \quad -\lambda^3 + 18\lambda^2 - 45\lambda = 0$$

$$\Rightarrow \quad -\lambda(\lambda^2 - 18\lambda + 45) = 0$$

$$\Rightarrow \quad -\lambda(\lambda-15)(\lambda-3) = 0$$

$$\Rightarrow \quad \lambda = 0, 3, 15$$

Case 1 : Suppose $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be Eigen vector corresponding to Eigen value $\lambda - 0,$ then

$$[A - 0, l]X_1 = 0$$

$$\Rightarrow \qquad \begin{bmatrix} 8-0 & -6 & 2 \\ -6 & 7-0 & -4 \\ 2 & -4 & 3-0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \qquad \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Since all rows are non identical then taking R_1 and R_2 rows as

Taking, $\quad 8.x - 6.y + 2.z = 0$

$$-6.x + 7.y - 4.z = 0$$

$$\therefore \qquad \frac{x}{24-14} = \frac{y}{-12+32} = \frac{z}{56-36}$$

$$\Rightarrow \qquad \frac{x}{10} = \frac{y}{20} = \frac{z}{20}$$

i.e., $\quad \frac{x}{1} = \frac{y}{2} = \frac{z}{2}$

$$\therefore \qquad X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Case 2 : Suppose $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be Eigen vector corresponding to Eigen value $\lambda - 3,$ then

$$[A - 3, l]X_1 = 0$$

$$\Rightarrow \qquad \begin{bmatrix} 8-3 & -6 & 2 \\ -6 & 7-3 & -4 \\ 2 & -4 & 3-3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$\Rightarrow \qquad \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Since all rows are non identical then taking R_1 and R_2 rows as

Taking, $5.x - 6.y + 2.z = 0$

$$-6.x + 4.y - 4.z = 0$$

$\therefore$ $$\frac{x}{24-8} = \frac{y}{-12+20} = \frac{z}{20-36}$$

$\Rightarrow$ $$\frac{x}{16} = \frac{y}{8} = \frac{z}{-16}$$

i.e., $$\frac{x}{2} = \frac{y}{1} = \frac{z}{-2}$$

$\therefore$ $$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

Case 3 : Suppose $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be Eigen vector corresponding to Eigen value $\lambda - 15,$ then

$$[A - 15, l]X_1 = 0$$

$\Rightarrow$ $$\begin{bmatrix} 8-15 & -6 & 2 \\ -6 & 7-15 & -4 \\ 2 & -4 & 3-15 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$\Rightarrow$ $$\begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

Since all rows are non identical then taking R_1 and R_2 rows as

Taking, $-7.x - 6.y + 2.z = 0$

$$-6.x + 8.y - 4.z = 0$$

$\therefore$ $$\frac{x}{24+16} = \frac{y}{-12-28} = \frac{z}{56-36}$$

$\Rightarrow$ $$\frac{x}{40} = \frac{y}{-40} = \frac{z}{-20}$$

i.e., $$\frac{x}{2} = \frac{y}{-2} = \frac{z}{-1}$$

$\therefore$ $$X_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$$

(b) Verify Cayley-Hamilton theorem for the following matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Solution : Given the matrix is

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

The characteristics equation is,

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $(2-\lambda)[(2-\lambda)(2-\lambda)-1]+1[-1.(2-\lambda)+1]+1[1-1(2-\lambda)]=0$

$\Rightarrow$ $(2-\lambda)[4+\lambda^2-4\lambda-1]+(\lambda-1)+(\lambda-1)=0$

$\Rightarrow$ $(2-\lambda)[\lambda^2-4\lambda+3]+2(\lambda-1)=0$

$\Rightarrow$ $2\lambda^2-8\lambda+6-\lambda^3+4\lambda^2-3\lambda+2\lambda-2=0$

$\Rightarrow$ $-\lambda^3+6\lambda^2-9\lambda^2+4=0$

$\Rightarrow$ $\lambda^3-6\lambda^2+9\lambda^2-4=0$ …….. (1)

This is required characteristic equation.

Eigen Values:

Clearly $\lambda = 1$ satisfy to equation (1), then $\lambda = 1$ is a root of equation (1).

Now, $\lambda^2(\lambda-1)-5\lambda(\lambda-1)+4(\lambda-1)=0$

$$(\lambda-1)(\lambda^2-5\lambda+4)=0$$

$$(\lambda-1)(\lambda-1)(\lambda-4)=0$$

$\Rightarrow$ $\lambda = 4,1,1$

Verification of Cayley-Hamilton theorem

By Cayley- Hamilton theorem every characteristic equation satisfy its characteristics equation, then from (1), we

Have

$$A^3 - 6A^2 + 9A^2 - 4I = 0 \qquad \text{.........(2)}$$

Now, $A^2 = A.A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} . \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

and $A^3 = A^2.A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$

$L.H.S. = A^3 - 6A^2 + 9A^2 - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6 \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 = R.H.S.$$

Hence verify Cayley-Hamilton theorem.

3. (a) Solve $\frac{dy}{dx} + y - 1$

Solution : Given LDE is,

$$\frac{dy}{dx} + y = 1 \qquad \text{.........(1)}$$

Here, P = 1 and Q = 1

$\therefore \quad I.F. = e^{\int Pdx} = e^{\int tdx} = e^x$

The solution is

$$y.(I.F.) = c + \int Q.(I.F.)dx$$

$\Rightarrow \quad ye^x = c + \int 1.e^x dx$

$\Rightarrow \quad \boxed{ye^x = c + e^x}$ **Answer**

(b) Solve $(D^3 - 3D^2 + 4)y = 0$

Solution : Given differential equation is,

$$(D^3 - 3D^2 + 4)y = 0$$

The A.E. is,

$$m^3 - 3m^2 + 4 = 0$$

Clearly $m = -1$ will satisfying the equation, then

$$m^2(m+1) - 4m(m+1) + 4(m+1) = 0$$

$\Rightarrow \quad (m+1)(m^2-4m+4)=0$

$\Rightarrow \quad (m+1)(m-2)^2=0$

$\Rightarrow \quad m=-1,2,2$

The complete solution is,

y = C.F.

$\Rightarrow \quad \boxed{y=c_1e^{-x}+(c_2+xc_3)e^{2x}}$ **Answer**

4. (a) Solve the exact differential equation $ye^x dx+(2y+e^x)dy=0$

Solution : Given differential equation is

$ye^x dx+(2y+e^x)dy=0$ ....(1)

Here, $M=ye^x$ and $N=2y+e^x$

Now, $\frac{\partial M}{\partial y}=e^x$ and $\frac{\partial N}{\partial x}=e^x$

$\Rightarrow \quad \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$

Therefore equation (1) is exact differential equation.

The solution of exact differential equation is,

$$\int_{y\text{ is contant}} \text{M dx} + \int_{\text{independent of } x} \text{N dx} = \text{c}$$

$\Rightarrow \quad \int ye^x dx+\int 2y\,dy=c$

$\Rightarrow \quad \boxed{ye^x+y^2=c}$ **Answer**

(b) Solve : $\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=e^{-x}$

Solution : Given : $(D^2+D+1)y=e^{-x}$

The A.E. is

$m^2+m+1=0$

$\Rightarrow \quad m=\frac{-1\pm\sqrt{1^2-4.1.1}}{2(1)}=\frac{-1\pm\sqrt{3}i}{2}$

$\Rightarrow \quad m=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$

$\therefore \quad C.F.=e^{-(1/2)x}\left[c_1\cos\left(\frac{\sqrt{3}}{2}x\right)+c_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$

Now, $P.I. = \frac{1}{D^2 + D + 1} e^{-x}$

$\Rightarrow \quad = \frac{1}{(-1)^2 - 1 + 1} e^{-x}$

$\Rightarrow \quad P.I. = e^{-x}$

The required solution is,

$$y = C.F. + P.I.$$

$$\Rightarrow \quad \boxed{y = e^{-(1/2)x}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right] + e^{-x}}$$ **Answer**

5. (a) Solve the simultaneous differential equations :

$$\frac{dx}{dt} = 2x + 6y \text{ and } \frac{dy}{dt} = x + y$$

Solution : Given differential equations are

$$\frac{dx}{dt} = 2x + 6y \qquad \text{............(1)}$$

and $$\frac{dy}{dt} = x + y \qquad \text{............(2)}$$

Let $D = \frac{d}{dt}$

$\therefore \quad (D-2)x - 6y = 0 \qquad \text{..............(3)}$

$-x + (D-1)y = 0 \qquad \text{..............(4)}$

Eliminate y from equation (3) and (4), we get

$$[(D-2)(D-1) - 6]x = 0$$

$\Rightarrow \quad (D^2 - 3D - 4)x = 0$

The A.E. is,

$$m^2 - 3m - 4 = 0$$

$\Rightarrow \quad (m-4)(m+1) = 0$

$\Rightarrow \quad m = 4, -1$

$\therefore \quad \boxed{x = c_1 e^{4t} + c_2 e^{-t}}$ **Answer**

Differentiate w.r.t. *t* we get

$$\frac{dx}{dt} = 4c_1 e^{4t} - c_2 e^{-t}$$

From equation (1), we get

$$6y = \frac{dx}{dt} - 2x$$

$\Rightarrow \quad 6y = 4c_1 e^{4t} - c_2 e^{-t} - 2\left(c_1 e^{4t} + c_2 e^{-t}\right)$

$\Rightarrow \quad 6y = 2c_1 e^{4t} - 3c_2 e^{-t}$

$\therefore \quad y = \frac{c_1}{3} e^{4t} - \frac{c_2}{2} e^{-t}$ **Answer**

(b) Solve $x\frac{d^2y}{dx^2} - (2x-1)\frac{dy}{dx} + (x-1)y = 0,$ **if** $y = e^x$ **is one integral.**

Solution : Given, $\frac{d^2y}{dx^2} + \left(-2 + \frac{1}{x}\right)\frac{dy}{dx} + \left(1 - \frac{1}{x}\right)y = 0$ ....(1)

Here, $P = -2 + \frac{1}{x}, Q = 1 - \frac{1}{x}$ and R = 0

Since $y_1 = e^x$, is a part of C.F., then Suppose the complete solution is

$$y = v\,y_1 = ve^x \qquad \text{.............(2)}$$

Where v is a function of x

Since $\frac{d^2v}{dx^2} + \left[P + \frac{2dy_1}{y_1 dx}\right]\frac{dv}{dx} = \frac{R}{y_1}$

$$\frac{d^2v}{dx^2} + \left[-2 + \frac{1}{x} + \frac{2}{e^x}\left(e^x\right)\right]\frac{dv}{dx} = 0$$

$\Rightarrow \quad \frac{d^2v}{dx^2} + \frac{1}{x}\frac{dv}{dx} = 0$

Taking, $z = \frac{dv}{dx} \Rightarrow \frac{dz}{dx} = \frac{d^2v}{dx^2}$

$\therefore \quad \frac{dz}{dx} + \frac{1}{x}z = 0$

$\Rightarrow \quad \frac{dz}{z} = -\frac{dx}{x}$

Integrating on both sides, we get

$$\log z = \log x = \log c_1$$

$\Rightarrow \quad z = \frac{c_1}{x}$

$\Rightarrow \quad \frac{dv}{dx} = \frac{c_1}{x}$

$\Rightarrow \qquad dv = c_1 \dfrac{dx}{x}$

Integrating on both sides, we get

$$v = c_1 \log x + c_2$$

Putting in equation (2), we get

$$y = [c_1 \log x + c_2] e^x$$ **Answer**

6. (a) Using method of removal of first derivative, solve the equation

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = e^x \sec x$$

Solution : Given the differential equation is,

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = e^x \sec x \qquad \ldots\ldots\ldots..(1)$$

Here, $P = -2 \tan x, Q = 5$ and $R = e^{'} \sec x$

Now this problem solve by Removable of first derivative method.

Suppose the complete solution is,

$$y = v\, y_1 \qquad \ldots\ldots\ldots..(2)$$

Where v is a function of x only.

Now we can find the value of y_1 such as

$$y_1 = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int (-2\tan x)\, dx} = e^{\log \sec x} = \sec x$$

and $Q_1 = Q - \dfrac{1}{4}p^2 - \dfrac{1}{2}\dfrac{dP}{dx} = 5 - \dfrac{1}{4}(-2\tan x)^2 - \dfrac{1}{2}\left(-2\sec^2 x\right) = 5 - \tan^2 x + \sec^2 x + 6$

and $R_1 = \dfrac{R}{y_1} = \dfrac{e^x \sec x}{\sec x} = e^x$

The normal form of Removable of first derivative is,

$$\frac{d^2v}{dx^2} + Q_2 v = R_1$$

$$\Rightarrow \qquad \frac{d^2v}{dx^2} + 6v = e^x \qquad \ldots\ldots\ldots\ldots..(3)$$

This is LDR of higher order.

The A.E. is

$$m^2 + 6 = 0$$

$\Rightarrow \qquad m = \pm\sqrt{6}i$

Therefore C.F.$= c_1 \cos\left(\sqrt{6}x\right) + c_2 \sin\left(\sqrt{6}x\right)$

Now, $P.I. = \dfrac{1}{D^2+6}e^x = \dfrac{1}{1^2+6}e^x = \dfrac{e^x}{7}$

The solution of equation (3) is,

$$v = C.F. + P.I. = c_1 \cos\left(\sqrt{6}x\right) + c_2 \sin\left(\sqrt{6}x\right) + \frac{e^x}{7}$$

Putting in equation (2), which our complete solution

$$\boxed{y = \left[c_1 \cos\left(\sqrt{6}x\right) + c_2 \sin\left(\sqrt{6}x\right) + \frac{e^x}{7}\right]\sec x}$$ **Answer**

(b) Using the method of variation of parameter, solve the equation:

$$\frac{d^2y}{dx^2} + y = \sec x$$

Solution : Given differential equation is

$$\frac{d^2y}{dx^2} + y = \sec x$$

Here $P = 0, Q = 1$ and $R = \sec x$

The A.E. is

$$m^2 + 1 = 0$$

$\Rightarrow \qquad m = \pm i$

Therefore C.F. is

$$y_c = c_1 \cos x + c_2 \sin x$$

Suppose $\qquad u = \cos x$ and $v = \sin x$

$\Rightarrow \qquad u' = -\sin x$ and $v' = \cos x$

and $\qquad w = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \left(\cos^2 x + \sin^2 x\right) = 1 \neq 0$

Suppose the complete solution of equation (1) is

$$y = Au + Bv = A(\cos x) + B(\sin x) \qquad \text{..............(2)}$$

Where A and B determine by the formula,

$$A = \int\left(-\frac{v.R}{w}\right)dx + c_1 = -\int \sin x(\sec x)dx + c_1$$

$\Rightarrow \qquad A = -\log \sec x + c_1$

and $\qquad B = \int \left(\frac{u.R}{w} \right) dx + c_2 = \int \cos x (\sec x) dx + c_2$

$\Rightarrow \qquad B = \int 1\, dx + c_2 = c + c_2$

Putting the values of A and B in equation (2), we get

$$y = [-\log \sec x + c_1] \cos x + [x + c_2] \sin x$$

$\Rightarrow \qquad y = c_1 \cos x + c_2 \sin x - \cos x (\log \sec x) + x \sin x$ **Answer**

7. (a) Use Lagrange's method solve the equation

$$x z p + y z q = x y$$

Solution : Given differential equation is …………(1)

$$x z p + y z q = x y$$

This is Lagrange PDE.

Here $P = x z, Q = y z \, and \, R = x y$

The Lagrange A.E. is

$$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}$$

Taking first two ratios, we get

$$\frac{dx}{xz} = \frac{dy}{yz} \Rightarrow \frac{dx}{x} = \frac{dy}{y}$$

Integrate both sides, we get

$$\log x = \log y + \log c_1$$

So that, $\qquad \frac{x}{y} = c_1$ ………..(2)

Taking last two ratios, we get

$$\frac{dy}{yz} = \frac{dz}{xy} \Rightarrow x\, dy = z\, dz$$

$\Rightarrow \qquad y c_1\, dy = z\, dz$ [From (2)]

Integrate both sides, we get

$$\frac{y^2}{2} c_1 = \frac{z^2}{2} + c_2 \Rightarrow \frac{y^2}{2} \left(\frac{x}{y} \right) = \frac{z^2}{2} + c_2$$

$\Rightarrow \qquad \frac{x\, y}{2} - \frac{z^2}{2} = c_2$

The General solution of equation (1), we get

$$\boxed{\phi\left[\frac{x}{y}, \frac{xy}{2} - \frac{z^2}{2}\right] = 0}$$ **Answer**

(b) **Solve** $\left(D^2 - DD' + 2D'^2\right)z = x + y$

Solution : Given PDE is,

$$\left(D^2 - DD' + 2D'^2\right)z = x + y \qquad \ldots\ldots\ldots\ldots(1)$$

The A.E. is

$$m^2 - m + 2 = 0$$

$$\Rightarrow \quad m = \frac{1 \pm \sqrt{(-1)^2 - 4(1)(2)}}{2(1)} = \frac{1 \pm \sqrt{-7}}{2}$$

$$\Rightarrow \quad m = \frac{1}{2} \pm \frac{\sqrt{7}}{2} i$$

$$\therefore \quad C.F. = \phi_1\left[y + \left(\frac{1}{2} + \frac{\sqrt{7}}{2}\right)x\right] + \phi_2\left[y + \left(\frac{1}{2} - \frac{\sqrt{7}}{2}\right)x\right]$$

Now, $P.I. = \dfrac{1}{D^2 - DD' + 2D'^2}(x + y)$

$$= \frac{1}{(1)^2 - (1)(1) + 2(1)^2} \iint v(dv)^2 \text{ where } v = x + y \text{ and } f(1,1) \neq 0 \text{ [By Short-cut Method]}$$

$$= \frac{1}{2}\left(\frac{v^3}{6}\right)$$

$$P.I. = \frac{(x + y)^3}{12}$$

The complete solution is,

$$z = C.F. + P.I.$$

$$\Rightarrow \quad \boxed{z = \phi_1\left[y + \left(\frac{1}{2} + \frac{\sqrt{7}}{2}\right)x\right] + \phi_2\left[y + \left(\frac{1}{2} - \frac{\sqrt{7}}{2}\right)x\right] + \frac{(x + y)^3}{12}}$$ **Answer**

8. (a) **Solve :** $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2$

Solution : Given PDE is,

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2$$

$\Rightarrow \quad \left[D^2 - 2DD' + D'^2 z = x^2 + xy + y^2\right]$ where $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$

The A.E. is,

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow \quad (m-1)^2 = 0$$

$$\Rightarrow \quad m = 1, 1$$

$$\therefore \quad C.F. = \phi_1(y+x) + x\phi_2(y+x)$$

Now, $P.I. = \frac{1}{D^2 - 2DD' + D'^2}\left(x^2 + xy + y^2\right)$

$$= \frac{1}{(D = D')^2}\left(x^2 + xy + y^2\right)$$

$$= \frac{1}{(D-D')^2}x^2 + \frac{1}{(D-D')^2}xy + \frac{1}{(D-D')^2}y^2$$

$$= \frac{1}{D^2\left[1 - \frac{D'}{D}\right]^2}x^2 + \frac{1}{D^2\left[1 - \frac{D'}{D}\right]^2}xy - \frac{1}{D'^2\left[1 - \frac{D}{D'}\right]^2}y^2$$

$$= \frac{1}{D^2}\left[1 - \frac{D'}{D}\right]^{-2}x^2 + \frac{1}{D^2}\left[1 - \frac{D'}{D}\right]^{-2}xy - \frac{1}{D'^2}\left[1 - \frac{D}{D'}\right]^{-2}y^2$$

$$= \frac{1}{D^2}\left[1 + 2\frac{D'}{D} + ...\right]x^2 + \frac{1}{D^2}\left[1 + 2\frac{D'}{D} + ...\right]xy - \frac{1}{D'^2}\left[1 + 2\frac{D}{D'} + ...\right]y^2$$

$$= \frac{1}{D^2}\left[x^2 + 2\frac{D'}{D}x^2 + ...\right] + \frac{1}{D^2}\left[xy + 2\frac{D'}{D}xy + ...\right] - \frac{1}{D'^2}\left[y^2 + 2\frac{D}{D'}y^2 + ...\right]$$

$$= \frac{1}{D^2}\left[x^2 + 0\right] + \frac{1}{D^2}\left[xy + 2\left(\frac{x^2}{2}\right)\right] - \frac{1}{D'^2}\left[y^2 + 0\right]$$

$$= \frac{1}{D}\left[\frac{x^3}{3}\right] + \frac{1}{D}\left[\frac{x^2 y}{2} + \frac{x^3}{3}\right] - \frac{1}{D'}\left[\frac{y^3}{3}\right] = \frac{x^4}{12} + \left[\frac{x^3 y}{6} + \frac{x^4}{12}\right] - \frac{y^4}{12}$$

$$P.I. = \frac{x^3 y}{6} - \frac{y^4}{12} + \frac{x^4}{6}$$

The Solution is,

$$\boxed{z = \phi_1(y+x) + x\phi_2(y+x) + \frac{x^3 y}{6} - \frac{y^4}{12} + \frac{x^4}{6}}$$

Answer

(b) **Solve :** $x^2 p + y^2 q = (x+y)z$

Solution : This is Lagrange LPDE of first order.

The A.E. is,

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y)z}$$

Taking, $$\frac{dx}{x^2} = \frac{dy}{y^2}$$

Integrating on both sides we get

$$-\frac{1}{x} = -\frac{1}{y} + c_1$$

$\Rightarrow$ $$\frac{1}{y} - \frac{1}{x} = c_1$$

Using the multiplier $\frac{1}{x}, \frac{1}{y}, -\frac{1}{z}$ respectively we get

$$\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z}$$

$$= \frac{x \quad y \quad z}{x + y - (x+y)}$$

$\Rightarrow$ $$= \frac{x \quad y \quad z}{0}$$

$\therefore$ $$\frac{dx}{x} + \frac{dy}{y} - \frac{dz}{z} = 0$$

Integrating on both sides we get

$$\log x + \log y - \log z = \log c_2$$

$\Rightarrow$ $$\log\left(\frac{x.y}{z}\right) = \log c_2$$

$\Rightarrow$ $$\frac{x.y}{z} = c_2$$

The general solution is,

$$\boxed{\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{x.y}{z}\right) = 0}$$ **Answer**

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