January : 2016 (CBCS)

Q.1 (a) If $y=\sin(m\sin^{-1}x)$,

Prove that $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+m^2y=0$

Sol. **Given :** The function $y=\sin\left(m\sin^{-1}x\right)$ …(i)

Differentiating equation (i) w.r.t. x, we get

$$y_1=\cos\left(m\sin^{-1}x\right)\frac{m}{\sqrt{1-x^2}}$$

$$(\sqrt{1-x^2})y_1=m\cos\left(m\sin^{-1}x\right).$$

Squaring both the sides, we get

$$(1-x^2)y_1^2=m^2\cos^2\left(m\sin^{-1}x\right)$$

$$\left(1-x^2\right)y_1^2=m^2\left[1-\sin^2\left(m\sin^{-1}x\right)\right]$$

$$\left(1-x^2\right)y_1^2=m^2\left[1-y^2\right] \quad \text{[From equation (i)]} \quad \text{…(ii)}$$

Differentiating equation (ii) w.r.t. x, we get

$$\left(1-x^2\right)2y_1y_2-2xy_1^2=-m^2 2yy_1$$

$$\left(1-x^2\right)y_2-xy_1=-m^2y$$

$$\boldsymbol{\left(1-x^2\right)y_2-xy_1+m^2y=0}.$$

Hence Proved.

Q.1 (b) The equation of the tangent at the point (2, 3) of the curve $y^2=ax^3+b$ is $y=4x-5$. Find the values of a and b.

Sol. **Given :** The equation of curve is,

$$y^2=ax^3+b \quad \text{…(i)}$$

Differentiating equation (i) w.r.t. x, we get

$$2y\frac{dy}{dx}=3ax^2$$

$$\frac{dy}{dx}=\frac{3ax^2}{2y}$$

$$\left(\frac{dy}{dx}\right)_{(2,3)}=\frac{3a(2)^2}{2(3)}=2a$$

The equation of tangent at (2, 3) is

$$(y-y_1)=\left(\frac{dy}{dx}\right)_{(2,3)}(x-x_1)$$

$$y-3=2a(x-2)$$

$$y=2ax-4a+3 \quad \text{…(ii)}$$

But given, the equation of tangent is

$$y=4x-5 \quad \text{…(iii)}$$

Equation (ii) and (iii) represent the same line hence comparing them, we get

$$\frac{1}{1}=\frac{2a}{4}=\frac{-4a+3}{-5}$$

$$\frac{2a}{4}=1 \Rightarrow a=2 \text{ and } \frac{-4a+3}{-5}=1 \Rightarrow a=2.$$

At the point (2, 3), from equation (i), we get

$$3^2 = 2(2)^3 + b \Rightarrow b = 9-16=-7.$$

$\therefore$ $\mathbf{a=2, b=-7}$. **Ans.**

Q.1 (c) Evaluate $\int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx$.

Sol. **Given :** $I = \int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx$

$$I = \int_0^{\pi/2} \frac{\sin 2x}{\sin^4 x + (1-\sin^2 x)^2}\, dx.$$

Putting $\sin^2 x = t$, so that $2\sin x \cos x\, dx = dt$ or $\sin 2x\, dx = dt$.

When $x=0$ then $t=0$ and when $x=\frac{\pi}{2}$ then $t=1$.

$$\therefore \quad I = \int_0^1 \frac{dt}{t^2+(1-t)^2} = \int_0^1 \frac{dt}{2t^2-2t+1}$$

$$I = \frac{1}{2}\int_0^1 \frac{dt}{t^2-t+\frac{1}{2}} = \frac{1}{2}\int_0^1 \frac{dt}{t^2-t+\frac{1}{4}-\frac{1}{4}+\frac{1}{2}}$$

$$I = \frac{1}{2}\int_0^1 \frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2} = \frac{1}{2}\cdot\frac{1}{\frac{1}{2}}\left[\tan^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$$

$$I = \left[\tan^{-1}(2t-1)\right]_0^1 = \tan^{-1}1 - \tan^{-1}(-1)$$

$$I = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$$

$\mathbf{I = \frac{\pi}{2}}$. **Ans.**

Q.2 (a) Expand by Maclaurin's theorem $e^{x\cos x}$ as far as the term x^3.

Sol. **Given :** The function $y = e^{x\cos x}$ $\Rightarrow (y)_0 = e^0 = 1$,

Differentiating y w.r.t. x successively, we get

$$y_1 = e^{x\cos x}(1\cdot\cos x - x\sin x) = y(\cos x - x\sin x) \quad \Rightarrow (y_1)_0 = (y)_0\cdot 1 = 1,$$

$$y_2 = y_1(\cos x - x\sin x) + y(-\sin x - 1\cdot\sin x - x\cos x)$$

$$y_2 = y_1(\cos x - x\sin x) - y(2\sin x + x\cos x) \quad \Rightarrow (y_2)_0 = (y_1)_0\cdot 1 = 1,$$

$$y_3 = y_2(\cos x - x\sin x) + y_1(-\sin x - 1\cdot\sin x - x\cos x)$$
$$-y_1(2\sin x + x\cos x) - y(2\cos x + 1\cdot\cos x - x\sin x)$$

$$y_3 = y_2(\cos x - x\sin x) - 2y_1(2\sin x + x\cos x) - y(3\cos x - x\sin x)$$
$$\Rightarrow (y_3)_0 = (y_2)_0\cdot 1 - (y)_0\cdot 3 = -2,$$

$$y_4 = y_3(\cos x - x\sin x) + y_2(-2\sin x - x\cos x) - 2y_2(2\sin x + x\cos x)$$
$$-2y_1(3\cos x - x\sin x) - y_1(3\cos x - x\sin x) - y(-4\sin x - x\cos x)$$

$$y_4 = y_3(\cos x - x\sin x) - 3y_2(2\sin x + x\cos x)$$
$$-3y_1(3\cos x - x\sin x) + y(4\sin x + x\cos x)$$
$$\Rightarrow (y_4)_0 = (y_3)_0 - 3(y_1)_0\cdot 3 = -11,$$

$$y_5 = y_4(\cos x - x\sin x) - 4y_3(2\sin x + x\cos x) - 6y_2(3\cos x - x\sin x)$$

$$+\,4y_1(4\sin x + x\cos x) + y(5\cos x - x\sin x)$$

$$\Rightarrow (y_5)_0 = (y_4)_0\cdot 1 - 0 - 6(y_2)_0\cdot 3 + 0 + (y)_0\cdot 5 = -24.$$

According to Maclaurin's series, we have

$$y = (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \ldots + \frac{x^n}{n!}(y_n)_0 + \ldots.$$

$$\mathbf{e^{x\cos x} = 1 + x + \frac{x^2}{2} - \frac{x^3}{3} - \frac{11x^4}{24} - \frac{x^5}{5} - \ldots\ldots\ldots\ldots}$$ **Ans.**

Q.2 (b) Prove that the curvature at the point (x, y) of the catenary $y = c\cos h\left(\frac{x}{x}\right)$ is $\frac{y^2}{c}$.

Sol. **Given :** The curve $y = c\cosh\left(\frac{x}{c}\right)$. …(i)

Differentiating equation (i) with respect to *x*, we get

$$\frac{dy}{dx} = c\sinh\left(\frac{x}{c}\right)\cdot\frac{1}{c} = \sinh\left(\frac{x}{c}\right).$$

Again differentiating with respect to *x*, we get

$$\frac{d^2y}{dx^2} = \frac{1}{c}\cosh\left(\frac{x}{c}\right).$$

∴ Radius of curvature

$$\rho = \frac{\left\{1 + (dy/dx)^2\right\}^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left\{1 + \sinh^2\left(\frac{x}{c}\right)\right\}^{3/2}}{\frac{1}{c}\cosh\left(\frac{x}{c}\right)} = c\frac{\left\{\cosh^2\left(\frac{x}{c}\right)\right\}^{3/2}}{\cosh\left(\frac{x}{c}\right)}$$

$$\rho = c\cosh^2\left(\frac{x}{c}\right) = c\left(\frac{y}{c}\right)^2$$ [Using equation (i)]

$$\boldsymbol{\rho = \frac{y^2}{c}}$$ **Hence Proved.**

Q.2 (c) Locate the stationary points of $x^4 + y^4 - 2x^2 + 4xy - 2y^2$ and determine their nature.

Sol. **Given :** $u = x^4 + y^4 - 2x^2 + 4xy - 2y^2$ …(i)

For maxima and minima of u, we must have

$$\frac{\partial u}{\partial x} = 4x^3 - 4x + 4y \quad \ldots(ii)$$

and $$\frac{\partial u}{\partial y} = 4y^3 + 4x - 4y \quad \ldots(iii)$$

Taking, $\frac{\partial u}{\partial x} = 0 \Rightarrow 4x^3 - 4x + 4y = 0$

$\Rightarrow$ $x^3 - x + y = 0$ …(iv)

and $\frac{\partial u}{\partial y} = 0 \Rightarrow 4y^3 + 4x - 4y = 0$

$\Rightarrow$ $y^3 + x - y = 0$ …(v)

Adding equation (iv) and (v), we get

$$x^3 + y^3 = 0 \Rightarrow (x + y)(x^2 - xy + y^2) = 0$$

$$x + y = 0 \text{ but } x^2 - xy + y^2 \neq 0$$

$$x = -y \quad \ldots(vi)$$

Putting in equation (ii), we get

$$x^3 - 2x = 0$$

$$x = 0, \quad x = \sqrt{2}$$

$\therefore$ $y = 0, \quad y = -\sqrt{2}$. [From equation (vi)]

Thus the required stationary points are $(0,0)$ and $(\sqrt{2}, -\sqrt{2})$.

Again partially differentiating equation (ii) w.r.t. x and y, we get

$$r = \frac{\partial^2 u}{\partial x^2} = 12x^2 - 4 \text{ and } s = \frac{\partial^2 u}{\partial x \partial y} = 4.$$

Partially differentiating equation (iii) w.r.t. y, we get

$$t = \frac{\partial^2 u}{\partial y^2} = 12y^2 - 4.$$

When $x = \sqrt{2}$, $y = -\sqrt{2}$, we have

$$r = 12(\sqrt{2})^2 - 4 = 20 > 0, \; s = 4 \text{ and } t = 12(-\sqrt{2})^2 - 4 = 20$$

$\therefore$ $rt - s^2 = (20)(20) - (4)^2 = 384 > 0$.

Therefore u is minimum at $(\sqrt{2}, -\sqrt{2})$.

When $x = 0$, $y = 0$, we have

and $r = 12(0)^2 - 4 = -4 < 0, \; s = 4$ and $t = 12(0)^2 - 4 = -4$

$\therefore$ $rt - s^2 = (-4)(-4) - (4)^2 = 16 - 16 = 0$.

The condition is doubtful and further investigation is needed.

Q.3 (a) If $u = \sec^{-1}\left(\frac{x^3 - y^3}{x + y}\right)$, then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u.$$

Sol. **Given :** Function $u = \sec^{-1}\left(\frac{x^3 - y^3}{x + y}\right)$

$$\therefore \quad \sec u = \frac{x^3 - y^3}{x + y} = \frac{x^3\left[1 - \left(\frac{y}{x}\right)^3\right]}{x\left[1 + \left(\frac{y}{x}\right)\right]} = \frac{x^2\left[1 - \left(\frac{y}{x}\right)^3\right]}{\left[1 + \left(\frac{y}{x}\right)\right]}$$

which is a homogeneous function of degree 2. Hence by Euler's theorem we have

$$x\frac{\partial}{\partial x}(\sec u) + y\frac{\partial}{\partial y}(\sec u) = 2\sec u$$

$$x\sec u \cdot \tan u\frac{\partial u}{\partial x} + y\sec u \cdot \tan u\frac{\partial u}{\partial y} = 2\sec u.$$

Dividing by $\sec u \cdot \tan u$ on both sides, we get

$$\boldsymbol{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\cot u}.$$

Hence Proved.

Q.3 (b) The radius of a sphere is found to be 10 cm with a possible error of 0.02 cm. What is the relative error in computing the volume?

Sol. **Given :** $r = 10$ cm and $\delta r = 0.02$ cm.

$\therefore$ Volume of sphere $= V = \frac{4}{3}\pi r^3$.

Taking log on both sides, we get

$$\log V = \log\left(\frac{4}{3}\right) + \log \pi + 3\log r \qquad \ldots(i)$$

Differentiating equation (i), we get

$$\frac{\delta V}{V} = 0 + 0 + 3\left(\frac{\delta r}{r}\right)$$

$$\therefore \qquad \frac{\delta V}{V} = \text{relative error in } V = 3\left(\frac{0.02}{10}\right) = 0.006\,.$$

Thus, relative error in volume of sphere is 0.006. **Ans.**

Q.3 (c) If $x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$, then show that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

Sol. **Given :** Functions $x = r\sin\theta\cos\phi,\ \ y = r\sin\theta\sin\phi$ and $z = r\cos\theta$.

By the definition of Jacobian, we have

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \sin\theta\cos\phi(0 + r^2\sin^2\theta\cos\phi) - r\cos\theta\cos\phi(0 - r\sin\theta\cos\phi\cos\theta)$$
$$+(-r\sin\theta\sin\phi)(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi$$
$$+\ r^2\cos^2\theta\sin^2\phi\sin\theta$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin^3\theta(\cos^2\phi + \sin^2\phi) + r^2\sin\theta\cos^2\theta(\cos^2\phi + \sin^2\phi)$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin^3\theta + r^2\sin\theta\cos^2\theta \qquad \left[\because\ \cos^2\theta + \sin^2\theta = 1\right]$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta\left[\sin^2\theta + \cos^2\theta\right]$$

$$\boldsymbol{\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta}\,.$$

Hence Proved.

Q.4 (a) Evaluate $\lim\limits_{n\to\infty}\left(\dfrac{1}{1+n^3} + \dfrac{4}{8+n^3} + \dfrac{9}{27+n^3} + \ldots\ldots + \dfrac{1}{2n}\right)$

Sol. **Given :** $I = \lim\limits_{n\to\infty}\left(\dfrac{1}{1+n^3} + \dfrac{4}{8+n^3} + \dfrac{9}{27+n^3} + \ldots.. + \dfrac{1}{2n}\right)$

The given series can be written as,

$$I = \lim_{n\to\infty}\left(\frac{1^2}{1^3+n^3} + \frac{2^2}{2^3+n^3} + \frac{3^2}{3^3+n^3} + \ldots.. + \frac{n^2}{n^3+n^3}\right)$$

The r^{th} term of the series is given by,

$$r^{th} \text{ term} = \frac{r^2}{n^3+r^3}, \text{ where } r \text{ varies from 1 to } n.$$

$\therefore$ The required limit of sum $= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{r^2}{n^3 + r^3}$

$$= \lim_{n\to\infty} \frac{1}{n^3} \sum_{r=1}^{n} \frac{r^2}{1+\left(\frac{r}{n}\right)^3} = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{\left(\frac{r}{n}\right)^2}{1+\left(\frac{r}{n}\right)^3}$$

For the corresponding definite integral, we have

Lower limit $= \lim_{n\to\infty}\left(\frac{r}{n}\right)$ for the first term

Lower limit $= \lim_{n\to\infty}\left(\frac{1}{n}\right)$ [$\because r = 1$ for the first term]

i.e., Lower limit = 0.

Upper limit $= \lim_{n\to\infty}\left(\frac{r}{n}\right)$ for the last term

Upper limit $= \lim_{n\to\infty}\left(\frac{n}{n}\right)$ [$\because r = n$ for the last term]

i.e., Upper limit = 1.

By summation of series, we get

$$I = \int_0^1 \frac{x^2}{1+x^3}\,dx \qquad \left[\because \frac{r}{n} = x \text{ and } \frac{1}{n} = dx\right]$$

Putting $x^3 = t$, so that $x^2 dx = \frac{dt}{3}$

$$I = \frac{1}{3}\int_0^1 \frac{dt}{t+1} = \frac{1}{3}\left[\log(t+1)\right]_0^1$$

$$I = \frac{1}{3}\left[\log 2 - 0\right]$$

$$\boldsymbol{I = \frac{1}{3}\log 2.}$$ **Ans.**

Q.4 (b) Prove that $\int_{-\infty}^{\infty} e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{a}$, $a > 0$.

Sol. **Given :** $I = \int_{-\infty}^{\infty} e^{-a^2x^2}\,dx$

$$I = 2\cdot\int_0^{\infty} e^{-a^2x^2}\,dx \qquad \ldots\text{(i)}$$

Putting $a^2x^2 = y$, i.e., $x = \frac{\sqrt{y}}{a}$, so that $dx = \frac{dy}{2a\sqrt{y}}$, from equation (i), we get

$$I = 2\int_0^{\infty} e^{-y}\,\frac{1}{2a\sqrt{y}}\,dy$$

$$I = \frac{1}{a}\int_0^{\infty} e^{-y}\,y^{-1/2}\,dy$$

$$I = \frac{1}{a}\int_0^{\infty} e^{-y}\,y^{\frac{1}{2}-1}\,dy = \frac{1}{a}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{a}$$

$$\boldsymbol{\int_{-\infty}^{\infty} e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{a}}.$$ **Hence Proved.**

Q.4 (c) Express $\int_0^1 x^m(1-x^n)^p\,dx$ in terms of beta functions and hence evaluate $\int_0^1 x^5(1-x^3)^{10}\,dx$.

Sol. **Given :** $I=\int_0^1 x^m(1-x^n)^p\,dx$

Putting $x^n=y$ i.e., $x=y^{1/n}$, so that $dx=\frac{1}{n}y^{\frac{1}{n}-1}dy$.

When $x=0$ then $y=0$ and when $x=1$ then $y=1$.

$$\therefore\quad \int_0^1 x^m(1-x^n)^p\,dx=\frac{1}{n}\int_0^1 y^{\frac{m}{n}}(1-y)^p\,y^{\frac{1}{n}-1}dy=\frac{1}{n}\int_0^1 y^{\frac{m+1}{n}-1}(1-y)^{(p+1)-1}dy$$

$$\boldsymbol{\int_0^1 x^m(1-x^n)^p\,dx=\frac{1}{n}\beta\left(\frac{m+1}{n},p+1\right)}\quad\left[\because\ \beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx\right]\quad\ldots\text{(i)}$$ **Ans.**

Putting $m=5, n=3$ and $p=10$ in equation (i), we get

$$\int_0^1 x^5(1-x^3)^{10}dx=\frac{1}{3}\beta\left(\frac{5+1}{3},10+1\right)=\frac{1}{3}\beta(2,11)=\frac{1}{3}\cdot\frac{\Gamma2\Gamma11}{\Gamma(2+11)}\ [\because\ \Gamma n=(n-1)!]$$

$$\int_0^1 x^5(1-x^3)^{10}dx=\frac{1}{3}\cdot\frac{1!\times10!}{12!}=\frac{1}{3}\cdot\frac{10!}{12\cdot11\cdot10!}$$

$$\boldsymbol{\int_0^1 x^5(1-x^3)^{10}dx=\frac{1}{396}}.$$ **Ans.**

Q.5 (a) Evaluate $\iint y\,dx\,dy$ over the part of the plane bounded by the line $y=x$ and the parabola $y=4x-x^2$.

Sol. **Given :** $I=\iint y\,dx\,dy$. …(i)

The region of integration R is bounded by curve,

$$y=x \quad\ldots\text{(ii)}$$

and

$$y=4x-x^2 \quad\ldots\text{(iii)}$$

Solving equation (ii) and (iii), we get

$$y=4x-x^2\ \Rightarrow\ x^2-3x=0$$

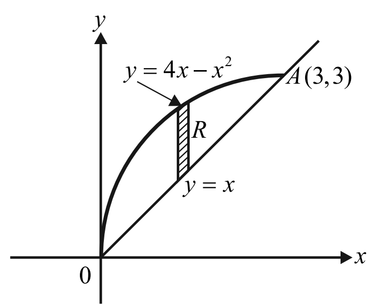
$$x=0,\quad x=3$$

$$\therefore\quad y=0,\quad y=3 \qquad \text{[From equation (ii)]}$$

Therefore, points of intersection of given curve are (0, 0) and (3, 3).

From the figure y varies from x to $4x-x^2$, whereas x varies from 0 to 3.

Hence the given double integral is,



$$I=\iint_R y\,dx\,dy=\int_{x=0}^{3}\int_{y=x}^{4x-x^2} y\,dy\,dx$$

$$I=\int_{x=0}^{3}\left[\frac{y^2}{2}\right]_{y=x}^{4x-x^2}dx$$

$$I=\frac{1}{2}\int_{x=0}^{3}\left[(4x-x^2)^2-(x)^2\right]dx$$

$$I=\frac{1}{2}\int_{x=0}^{3}[15x^2+x^4-8x^3]\,dx$$

$$I=\frac{1}{2}\left[5x^3+\frac{x^5}{5}-2x^4\right]_{x=0}^{3}=\frac{1}{2}\left[405+\frac{243}{5}-162\right]$$

$$I=\frac{54}{5}$$

$$\boldsymbol{\iint y\,dx\,dy=\frac{54}{5}}.$$ **Ans.**

Q.5 (b) Evaluate $\int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\;dz\;dy\;dx.$

Sol. **Given :** $I=\int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx$

$$I=\int_{x=0}^1\int_{y=0}^{1-x} xy\left[\int_{z=0}^{1-x-y} z\,dz\right]dy\,dx$$

$$I=\int_{x=0}^1\int_{y=0}^{1-x} xy\left[\frac{z^2}{2}\right]_{z=0}^{1-x-y} dy\,dx$$

$$I=\int_{x=0}^1\int_{y=0}^{1-x} xy\,\frac{(1-x-y)^2}{2}\,dy\,dx$$

$$I=\frac{1}{2}\int_{x=0}^1 x\left\{\int_{y=0}^{1-x} y\left[(1-x)^2-2(1-x)y+y^2\right]dy\right\}dx$$

$$I=\frac{1}{2}\int_{x=0}^1 x\left\{\int_{y=0}^{1-x}\left[(1-x)^2 y-2(1-x)y^2+y^3\right]dy\right\}dx$$

$$I=\frac{1}{2}\int_{x=0}^1 x\left[(1-x)^2\frac{y^2}{2}-2(1-x)\frac{y^3}{3}+\frac{y^4}{4}\right]_{y=0}^{1-x}dx$$

$$I=\frac{1}{2}\int_{x=0}^1 x\left[\frac{(1-x)^4}{2}-2\frac{(1-x)^4}{3}+\frac{(1-x)^4}{4}\right]dx$$

$$I=\frac{1}{24}\int_{x=0}^1 x(1-x)^4\,dx$$

Putting $1-x=t$, so that $dx=-dt$

$$I=\frac{1}{24}\int_{t=1}^0 (1-t)t^4(-dt)=\frac{1}{24}\int_{t=0}^1 (t^4-t^5)\,dt$$

$$I=\frac{1}{24}\left[\frac{t^5}{5}-\frac{t^6}{6}\right]_{t=0}^1=\frac{1}{24}\left[\frac{1}{5}-\frac{1}{6}\right]=\frac{1}{24}\left[\frac{6-5}{30}\right]$$

$$\int_0^1\int_0^{1-x}\int_0^{1-x-y} xyz\,dz\,dy\,dx=\frac{1}{720}.$$ **Ans.**

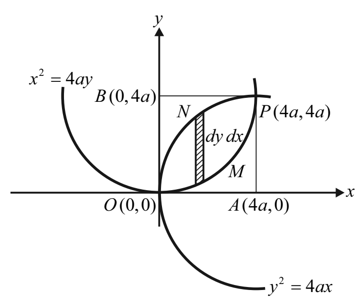
Q.5 (c) Find the area enclosed by the parabolas $y^2=4ax$ and $x^2=4ay$.

Sol. **Given :** The equations of parabolas are

$$y^2=4ax \quad \ldots(i)$$

and $$x^2=4ay \quad \ldots(ii)$$

Squaring both sides in equation (ii), we get



$$x^4=16a^2y^2$$

$$x^4=16a^2(4ax)$$

$$x(x^3-64a^3)=0$$

$$x=0 \text{ and } x^3=64a^3$$

$$x=0 \text{ and } x=4a.$$

Putting in equation (i), we get

$$y=0 \text{ and } y=4a.$$

∴ Required point of intersection are $(0,0)$ and $(4a,4a)$.

Here

(i) y varies from $\frac{x^2}{4a}$ to $\sqrt{4ax}$.

(ii) x varies from 0 to $4a$.

∴ Required area is,

$$A=\iint dx\,dy$$

$$A = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{\sqrt{4ax}} dx\, dy$$

$$A = \int_{x=0}^{4a} [y]_{y=\frac{x^2}{4a}}^{\sqrt{4ax}} dx = \int_{x=0}^{4a} \left[\sqrt{4ax} - \frac{x^2}{4a}\right] dx$$

$$A = 2\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$A = \frac{32}{3}a^2 - \frac{16}{3}a^2$$

The required area is $A = \frac{16}{3}a^2$ square units. Ans.

Q.6 (a) Evaluate $\int_a^b e^x\, dx$ as limit of sum.

Sol. **Given :** $f(x) = e^x$.

We know that by definition of definite integral as limit of sum,

$$\int_a^b f(x)\, dx = \lim_{h \to 0} h \sum_{r=0}^{n-1} f(a + rh)$$

$$\int_a^b e^x\, dx = \lim_{h \to 0} h \sum_{r=0}^{n-1} e^{a+rh}$$

$$\int_a^b e^x dx = \lim_{h \to 0} h\left[e^a + e^{a+h} + e^{a+2h} + e^{a+3h} + ... + e^{a(n-1)h}\right]$$

$$\int_a^b e^x dx = \lim_{h \to 0} h\, e^a \left[1 + e^h + e^{2h} + ... + e^{(n-1)h}\right]$$

$$\int_a^b e^x dx = \lim_{h \to 0} h\, e^a \left[\frac{1(e^{nh} - 1)}{e^h - 1}\right] \qquad \left[\because S_n = \frac{a(r^n - 1)}{r - 1}, r > 1\right]$$

$$\int_a^b e^x dx = \lim_{h \to 0} e^a (e^{nh} - 1) \lim_{h \to 0}\left(\frac{h}{e^h - 1}\right)$$

$$\int_a^b e^x dx = e^a (e^{b-a} - 1) \lim_{h \to 0} \frac{\frac{d}{dh}(h)}{\frac{d}{dh}(e^h - 1)} \qquad [\because nh = b - a]$$

[Using L' Hospital's rule]

$$\int_a^b e^x dx = (e^b - e^a). \lim_{h \to 0} \frac{1}{e^h}$$

$$\boldsymbol{\int_a^b e^x\, dx = e^b - e^a}.$$ Ans.

Q.6 (b) Express in terms of the gamma function : $\int_0^\infty x^n e^{-a^2 x^2} dx.$

Sol. **Given :** $I = \int_0^\infty x^n e^{-k^2 x^2} dx$

Putting $x^2 = t$, i.e., $x = t^{\frac{1}{2}}$, so that $dx = \frac{1}{2} t^{\frac{-1}{2}} dt$, we get

$$I = \int_0^\infty x^n e^{-k^2 x^2}\, dx = \int_0^\infty t^{n/2} e^{-k^2 t}\, \frac{1}{2} t^{-1/2}\, dt$$

$$I = \frac{1}{2}\int_0^\infty t^{\left(\frac{n-1}{2}\right)} e^{-k^2 t}\, dt \qquad \text{where } c = k^2 \text{ and } m = \frac{n+1}{2}$$

$$I = \frac{1}{2}\int_0^\infty t^{m-1} e^{-ct}\, dt = \frac{1}{2}\frac{\Gamma m}{c^m} \qquad \left[\because \int_0^\infty e^{-cy} y^{n-1} dy = \frac{\Gamma n}{c^n}\right]$$

$$I = \frac{\Gamma m}{2k^{2m}} \qquad \left[\because c = k^2\right]$$

$$\mathbf{I = \frac{1}{2k^{n+1}}\Gamma\left(\frac{n+1}{2}\right)}. \qquad \left[\because m = \frac{n+1}{2}\right]$$

Hence Proved.

Q.6 (c) Change the order of integration in $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the same.

Sol. Given : $I = \int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dx\,dy$ …(i)

We draw the bounded region from the given curves :

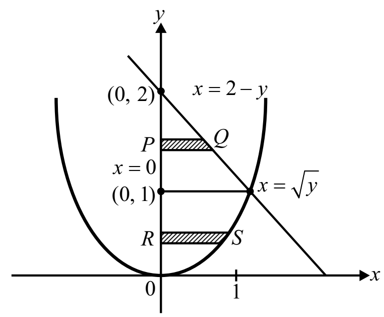
$x = 0, x = 1, y = x^2$ and $y = 2 - x$,

The possible points for bounded region are : (0,0) (1,1) and (0,2).

On changing the of order of integration, integrate first w.r.t. x by taking two strips parallel to x-axis say, PQ and RS.

Limits :

1. x varies from $R(x=0)$ to $S(x=\sqrt{y})$ and y varies from $y=0$ to $y=1$.
2. x varies from $P(x=0)$ to $Q(x=2-y)$ and y varies from $y=0$ to $y=2$.



$\therefore$
$$I = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$I = \int_{y=0}^{1}\left[\frac{x^2}{2}\right]_0^{\sqrt{y}} y\,dy + \int_{y=1}^{2}\left[\frac{x^2}{2}\right]_0^{2-y} y\,dy$$

$$I = \frac{1}{2}\int_0^1 y^2 dy + \frac{1}{2}\int_1^2 y(2-y)^2\,dy$$

$$I = \frac{1}{2}\left[\frac{y^3}{3}\right]_0^1 + \frac{1}{2}\left[\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$\mathbf{I = \int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dx\,dy = \int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy + \int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy = \frac{3}{8}}.$$

Ans.

Q.7 (a) Verify Rolle's theorem, where $f(x) = 2x^3 + x^2 - 4x - 2$.

Sol. Given : The function $f(x) = 2x^3 + x^2 - 4x - 2$

Since a polynomial function is everywhere continuous and differentiable, so the given function is continuous as well as differentiable in every interval.

To identify the interval, we first solve the equation, $f(x) = 0$.

i.e.,
$$2x^3 + x^2 - 4x - 2 = 0$$
$$x^2(2x+1) - 2(2x+1) = 0$$
$$(x^2 - 2)(2x+1) = 0$$
$$x^2 = 2 \text{ or } x = -\frac{1}{2}$$
$$x = \pm\sqrt{2} \text{ or } x = -\frac{1}{2}.$$

So, we consider the given function in $[-\sqrt{2}, \sqrt{2}]$.

Clearly, $f(-\sqrt{2}) = f(\sqrt{2}) = 0$.

Thus, all the conditions of Rolle's theorem are satisfied. So there must exist atleast one point $c \in (-\sqrt{2}, \sqrt{2})$ such that $f'(c) = 0$.

But, $f'(x) = 6x^2 + 2x - 4$

$\therefore \quad f'(c) = 0 \Rightarrow 6c^2 + 2c - 4 = 0$

$2(3c-2)(c+1) = 0$

$c = \frac{2}{3}$ or $c = -1$.

Clearly, both these points lie in $(-\sqrt{2}, \sqrt{2})$.

Hence, Rolle's theorem is verified. **Hence Proved.**

Q.7 (b) If $u = f(y-z, z-x, x-y)$, prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.

Sol. **Given :** Function $u = f(y-z, z-x, x-y)$.

Let $X = y - z,\ Y = z - x$ and $Z = x - y$ …(i)

Then $u = f(X, Y, Z)$, where each one of X, Y, Z is a function of x, y, z.

Partially differentiating equation (i) w.r.t. x, y and z respectively, we get

$$\frac{\partial X}{\partial x} = 0, \frac{\partial X}{\partial y} = 1, \frac{\partial X}{\partial z} = -1,$$

$$\frac{\partial Y}{\partial x} = -1, \frac{\partial Y}{\partial y} = 0, \frac{\partial Y}{\partial z} = 1,$$

and $$\frac{\partial Z}{\partial x} = 1, \frac{\partial Z}{\partial y} = -1, \frac{\partial Z}{\partial z} = 0 .$$

Now $$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial x} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial x} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial X}\cdot(0) + \frac{\partial u}{\partial Y}\cdot(-1) + \frac{\partial u}{\partial Z}\cdot(1) = -\frac{\partial u}{\partial Y} + \frac{\partial u}{\partial Z} \quad \text{…(ii)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial y} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial y} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial y}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial X}\cdot(1) + \frac{\partial u}{\partial Y}\cdot(0) + \frac{\partial u}{\partial Z}\cdot(-1) = \frac{\partial u}{\partial X} - \frac{\partial u}{\partial Z} \quad \text{…(iii)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\cdot\frac{\partial X}{\partial z} + \frac{\partial u}{\partial Y}\cdot\frac{\partial Y}{\partial z} + \frac{\partial u}{\partial Z}\cdot\frac{\partial Z}{\partial z}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial X}\cdot(-1) + \frac{\partial u}{\partial Y}\cdot(1) + \frac{\partial u}{\partial Z}\cdot(0) = -\frac{\partial u}{\partial X} + \frac{\partial u}{\partial Y} \quad \text{…(iv)}$$

Adding equations (ii), (iii) and (iv), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$ **Hence Proved.**

Q.7 (c) Trace the curve $y^2(2a - x) = x^3$.

Sol. **Given :** The equation of curve

$$y^2(2a - x) = x^3 \quad \text{…(i)}$$

The tracing of curve have following steps :

(i) **Symmetry :** Here in equation (i) all power of y are even, hence the curve is symmetrical about the x-axis.

(ii) **Origin :** There is no constant term in this equation. By putting $x = 0$, we have $y = 0$ the curve passes through the origin.

The tangents at the origin are $y = 0$. [Equating to zero the lowest degree terms.]

$\therefore$ Origin is a **cusp.**

(iii) **Points of intersection :**

When $x = 0$ then $y = 0$.

When $y = 0$ then $x = 0$.

i.e., the curve meets the co-ordinate axis only at origin.

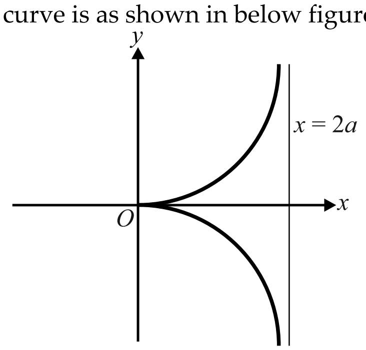
(iv) **Asymptotes :** Equating coefficient of higher power of x and y to 0. We have the asymptotes as follows.

The curve has an asymptote $x = 2a$ (parallel to y- axis).

(v) **Region :** We have, $y^2 = x^3/(2a - x) \Rightarrow y = \sqrt{\dfrac{x^3}{2a - x}}$.

When x is – ve, y^2 is – ve (i.e. y is imaginary) so that no portion of the curve lies to the left of the y-axis. Also when $x > 2a, y^2$ is again – ve, so that no portion of the curve lies to the right of the line $3x = 2a$.

Hence the shape of the curve is as shown in below figure. This curve is known as **cissoids**.



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